Innovation and Welfare Gains from Trade: do Small Firms Matter?

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PRELIMINARY AND INCOMPLETE

Abstract

We explore how innovation affects aggregate welfare gains from trade. While it is well known that exporters invest in productivity during trade liberalizations ("innovate"), it is not clear how this affects aggregate welfare. To study this, we build a model of international trade with innovation and derive analytically the firm's size distribution in equilibrium. We then assess the impact of reductions in trade costs. Entrants (small firms) play a key role: the larger the entrants, the smaller the change in aggregate innovation. Comparing steady states, a 1% reduction in iceberg costs increases productivity between 0.17% and 1.30%, depending on the relative size of entrants. To assess the impact on welfare, we compute analytically the transition between steady states. The discounted welfare gains are 10 times larger than models without innovation. This sharply contrasts previous research that finds that changes in entry levels cancel out the potential effects of innovation. We argue that the way entrants enter matters, and the reason for the previous results is that entrants were modeled as counterfactually large firms.

1 Introduction

Trade liberalization brings about a series of changes in the industrial organization of the liberalized country or sector that affects the distribution of firms. First, the increased size of the export market implies that exporters become relatively larger. This has led most studies to focus more on the upper tail of the distribution, ignoring what happens at the bottom. But the bottom also changes: non exporters become exporters; and innovation affects firm size across the board. Several papers find that exporters have more incentives

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to innovate, while non exporters, faced with increased competition, have lower incentives to innovate when trade costs fall. All these changes have potentially large effects on welfare. To understand these effects on welfare, we first need to understand how the distribution of firms interplays with trade liberalization. In other words, the distribution of firms talks, it tells us how different policies affects the different firms in the economy.

This paper explores the link between trade liberalization, the firm size distribution, and welfare. To do this, we develop a model of international trade with innovation, endogenizing the firm size distribution, measured as number of employees or sales. We then study the effects of lowering trade costs on welfare and how the distribution of firms sheds light on the channels that generate those welfare gains.

The model builds on Melitz (2003), modified by adding process innovation. There are heterogeneous firms that make production, innovation and exporting decisions in every period, and may exogenously die with a fixed probability. There is a large pool of potential entrants that would become productive if the expected profits of being a incumbent is larger than an exogenously given fixed cost of entry. When a firm is born it gets a random initial level of productivity, thereafter it makes innovation decisions that lead to growth. The probability distribution of the initial level of productivity determines the distribution of small firms (entrants), and is one of the main targets of this paper.

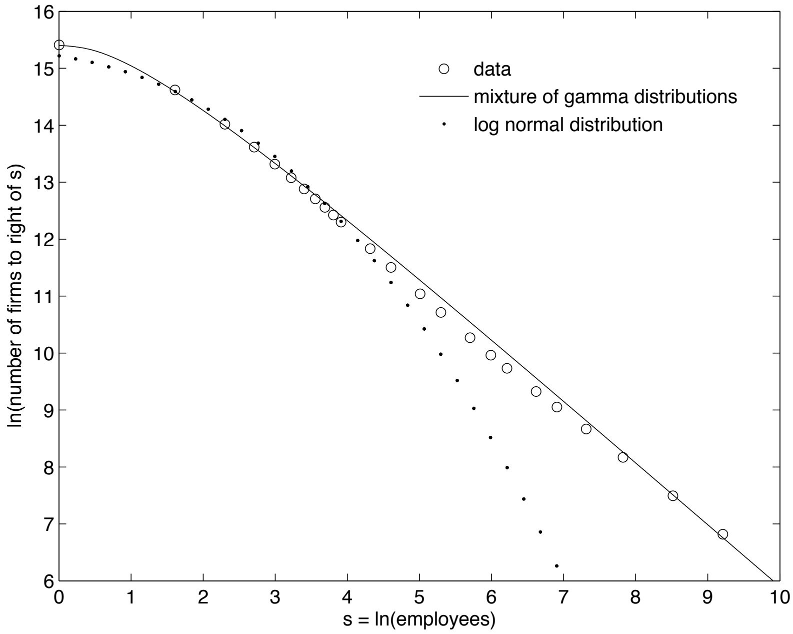
We design the model to replicate four well established facts of the industrial organization and international trade literature. These are: (i) Gibrat's law: firm growth rates are independent of firm size; (ii) Zipf's law: the upper tail of the firm size distribution is close to a Pareto distribution; (iii) exporters are larger than non exporters; and (iv) exporters grow faster than non exporters. The first two facts are well established in the industrial organization literature. The latter two emerge consistently from empirical papers relating firms and international trade, starting with Bernard and Jensen (1999).

These four facts are incorporated via the innovation cost function and the existence of a fixed export cost, as in Melitz (2003). We propose an innovation cost function so that in equilibrium the rate of growth for a given firm is independent of its size (Gibrat's law). Also, the assumption of a sunk export cost implies that, in equilibrium, only large firms choose to export.

The other two facts emerge in equilibrium. From Gabaix (2011) we know Gibrat's law implies Zipf's law. That is, the equilibrium firm size distribution has a Pareto upper tail for a very wide range of entrant distributions. We derive this relationship in closed form solution, as in Acemoglu and Cao (2010), Luttmer (2010), and Benhabib et al. (2011). The second equilibrium outcome is that exporters grow faster than non exporters. Since exporters face larger demands, they choose to invest more in innovation, and therefore grow faster.

Our equilibrium firm size distribution is a particular version of Zipf's law. Zipf's law characterizes the upper tail of the distribution as close to a Pareto distribution. While this is true in our equilibrium, we can go one step further and describe the middle part of the distribution as also close to Pareto, with a different curvature parameter.

Figure 1: The Firm Size Distribution in US (Luttmer, 2007)



We illustrate this finding by using the plot in Luttmer (2007) to describe the firm size distribution in the data, replicated in Figure 1. This plot shows the log of the number of employees (x) in the horizontal axis, and the log of the number of firms with more than x employees on the vertical one. The dots represent the data, the solid line is an approximation with a straight line. The straight line approximation is quite accurate, which is a characteristic of the Pareto distributions.

We show this same plot with model implications in Figure 2. The distribution is a combination of two straight lines. For relatively small values of x, the slope is steeper than for larger values. By paying close attention to the plot in Figure 1 we uncover a similar pattern. It becomes clear that the dots are initially on top of the straight line, then fall below, and then eventually rise to a level above the straight line.

Figure 2: The Firm Size Distribution in our Model

The slope of the distribution is determined by the rate of growth of firms in that segment of the distribution. Since exporters grow faster, the slope for large firms is flatter. We are not the first to notice that the slope for exporters is flatter than the slope for non exporters. di Giovanni et al. (2010) observe this empirically by analyzing the distribution of firms separating exporters from non exporters. We add theoretical validity to their empirical observation.

We next turn to studying the effects of a drop in trade costs, and the role of the distribution in determining the welfare gains. To study welfare gains, we split the gains into direct and indirect gains, following Atkeson and Burstein (2010). Direct gains are the gains that occur directly from the removal of a distortion (the trade costs), fixing all firm decisions. The indirect effect takes into account all the changes in firm decisions.

By comparing steady states, we find that the magnitude of welfare gains, and in particular the indirect effect, heavily depend on the distribution of small firms. If entrants (more likely to be in the lower tail of the distribution) enter with relatively low productivity technologies, then the gains of reducing trade costs are larger. This increase is due to an increase in the innovation decisions and firm entry (the *indirect* effect, following Atkeson and Burstein (2010)). By construction, the *direct* effect (the one occurring only because of the removal of a distortion such as the trade costs) is constant independently of the distribution of entrants.

We also compute the transition to the new steady state in closed form. We focus on a special case of our model to compute the transition, in which all entrants enter with equal productivity. We use this transition path to compute the welfare gains from trade: the equivalent variation in consumption from a change in marginal trade costs, defined as the change in consumption at the old steady state that leaves households indifferent between the old steady state and the transition to the new steady state. We find that even taking into account the transition to the new steady state, the effects of innovation on the welfare gains from trade are huge.

This is an important message to those papers that focus only on the upper tail. Papers such as Gabaix (2011) and di Giovanni and Levchenko (2009) conclude that the behavior at the top is what really matters for aggregate outcomes. These studies assume that firm size is exogenous. The fact that the distribution of firms is fat tailed implies that aggregate outcomes are more heavily affected by the shocks to large firms. But with innovation, firm size is endogenous, and a change in trade costs has effects on the size of every firm, both large (exporters) and small (non exporters).

The intuition of why small firms matter stems from the free entry condition. To see this, suppose all entrants immediately become exporters, as in Krugman (1980). A drop in trade costs will increase expected profits from entry considerably, and many firms will want to enter, making entry extremely elastic. But the free entry condition pins down the expected profits from entry. Entry restores the expected profits for an entrant to their previous levels, thus restoring the old equilibrium. If profits do not increase, neither will innovation and welfare will not depend on innovation. On the other hand, if entrants are very unlikely to become exporters, entry is inelastic with respect to trade costs, and the gain perceived by exporters will not be counteracted by an increase in costs, and innovation will increase, producing welfare gains and changes in the firm size distribution.

The importance of the free entry condition was first discussed in Atkeson and Burstein (2010). They argue that in many cases, the free entry condition implies that innovation does not increase welfare gains from trade. These cases are three: when all firms export as in Krugman (1980); when exporting is inelastic and the interest rate is zero; and when innovation is inelastic and the interest rate is zero, as in Melitz (2003). Our model delivers the same result as theirs for these three cases. Also, for the other cases they analyze, we find similar effects on welfare and the contribution of innovation to welfare increases. These cases relax the assumption of inelastic exporting, inelastic innovation, and zero interest rate. The authors conclude that innovation does not add much to the welfare gains from trade because the indirect effect is very small relatively to the direct effect, and any increase via the indirect effect takes a long time to happen, and therefore it is quite small in present value.

The link between innovation and trade liberalization has been the subject of a vast recent literature. Rubini (2011) finds that adding innovation is key to account for productivity gains from trade, when productivity is measured as in the national accounts, and to account for the large trade elasticity observed by Head and Ries (2001) and Clausing (2001) during the Canada-U.S. Free Trade Agreement, while being consistent with micro, high frequency estimates of this trade elasticity. Examples of papers measuring this high frequency elasticity include estimates using macro models such as Corsetti et al. (2008), Backus et al. (1994), and Heathcote and Perri (2004), and micro estimates such as Broda and Weinstein (2006).

Empirically, many papers have found evidence of innovation strongly connected to trade liberalization. Firms that become exporters during a trade liberalization period exhibit sharp increases in innovation expense and productivity. Bustos (2011) finds that Argentine firms that became exporters during the Mercosur increased their innovation expense dramatically. De Loecker (2007) finds that Slovenian firms that entered the export market following Slovenia's independence increased their productivity by up to fifty percent relative to a control group of firms that did not enter the export market. Van Biesebroeck (2005) finds similar results for Sub-Saharan African firms. Lileeva (2008) and Lileeva and Trefler (2010) find that Canadian firms that entered the export market during the Canada-US Free Trade Agreement increased their investment in innovation relative to other firms.

2 The Model

The model builds on Melitz (2003). Time is continuous. There are two symmetric countries that produce a continuum of differentiated goods that can be traded. Each good can only be produced in one country. There is an infinitively lived representative consumer that derives utility from consuming as many goods as possible. There are incumbent firms each period that make production, innovation, and exporting decisions. Firms die each period with an exogenous probability δ. There is a pool of potential entrants that can enter by paying an entry cost κ_E.

With continuous time, the preferences of the consumer in country i are given by the following utility function, for $i = 1, 2$:

$$U_i\left(q_i(\omega, t)\right) = \int_0^{\infty} e^{-\rho t} \ln Q_i(t) dt$$

$$Q_i(t) = \left[\int_{\Omega_i(t)} q_i(\omega, t)^{\frac{\sigma-1}{\sigma}} d\omega + \int_{\Omega_i^*(t)} q_i(\omega, t)^{\frac{\sigma-1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}}$$

where ω is the name of the good consumed, Ω_i is the set of goods produced in country i and Ω_i^* is the set of goods produced in country $j \neq i$ and imported into country i. $\sigma > 1$ is the elasticity of substitution between goods.

Each instant, there is a continuum of incumbent firms that produce the goods. Firms are owned by the domestic consumer. Each firm is a monopolist producing each good. Given a productivity level z and labor services n, the firm producing good ω has access to the following technology:

$$y(\omega; z, n) = zn$$

A firm can make innovation expenses to increase its productivity level z. We choose a functional form for the innovation cost that guarantees that in equilibrium Gibrat's law emerges. That is, in equilibrium, firm growth rate is independent of firm size. The innovation cost is in labor units. The cost of increasing productivity by an amount $\dot{z}$ depends on the current productivity level z, and is given by:

$$c(z, \dot{z}) = \frac{\kappa_I z^{\sigma-1}}{2} \left(\frac{\dot{z}}{z}\right)^2$$

This cost function says that to increase productivity by a certain proportion, a firm must incur a cost proportional to that proportion squared. Additionally, if a very productive firm wants to increase its productivity by 10%, it must incur a cost that is greater than what a low productivity firm would need to incur to increase its productivity by 10%. This is why the term $z^{\sigma-1}$ appears in the cost function. The term $\sigma - 1$ in the exponent is useful for the solution to be in closed form. κ_I determines how costly innovation is.

As in Ruhl (2008), any firm can export by incurring a sunk export cost equal to κ_X units of labor. Once a firm becomes an exporter, it remains an exporter until it dies, without the need of paying additional export costs.

There is a large pool of potential entrants that can enter anytime by incurring an entry cost equal to κ_E units of labor. After paying the entry cost, entrants draw their productivity z from an exogenous distribution $f(z)$. It is worth to notice that, in equilibrium, younger firms are relatively smaller firms. This distribution function $f(z)$ shapes the distribution of entrants, and therefore the lower tail of the size distribution of firms. We assume $z \in [1, \infty)$.

Exports are subject to trade costs. We model these costs is two ways: as tariffs, and as iceberg transport costs. In the firms case, exporters produce the same amount as importers consume. If importers pay a price p to the exporter, additionally it pays a proportion τ of this price to its government. In turn, the government rebates this revenue lump sum to the consumer.

When we model trade costs as iceberg costs, we assume that transport depletes a proportion τ of the good. So if a consumer consumes an amount q of a good, the exporter must ship an amount $(1 + \tau)q$. Eliminating iceberg costs in this case is an improvement in the technology set. Eliminating tariffs is a policy experiment, going from a second best to a first best.

Both of these assumptions are common in the literature. Ruhl (2008) and Rubini (2011) assume trade costs are tariffs. Atkeson and Burstein (2010) and Melitz (2003) assume they are iceberg costs. Rubini (2011) discusses the theoretical difference between these two, arguing that these two assumptions can deliver important quantitative differences. In this paper we show that this is the case: the gains of reducing tariffs can double the gains of reducing iceberg costs.

The labor market clearing condition closes the model. Let $M(t)$ be the measure of entrants at time t. The labor market clearing condition is

$$1 = \int_{\Omega_i(t)} (n(\omega, t) + \bar{c}(\omega, t)) d\omega + M(t)\kappa_E \tag{1}$$

where $\bar{c}(\omega, t)$ is the labor demand for innovation of firm ω at time t.

3 Symmetric Equilibrium

We identify a monopolistically competitive symmetric equilibrium for this economy. The symmetry allows us to drop the subindex i from everywhere.

To solve for the equilibrium, we introduce prices. Let $w(t)$ be the wage rate at time t. We use this as numeraire, so set $w(t) = 1$ for all t. The price of good ω is $p(\omega)$. In equilibrium we show that the price before tariffs for an exported good is the same as the price of the same good sold domestically, so we do not introduce notation for the price of an exported good. This price is set by the monopolist to maximize profits subject to the demand for its product. This demand function comes from the consumer maximization problem. Each instant, consumers choose how much to consume of each good taking each price as given. In equilibrium, symmetry implies there is no borrowing and lending between countries, so the problem of the consumer becomes a static problem:

$$\max \ln Q(t)$$
$$s.t.$$
$$Q(t) = \left[\int_{\Omega(t)} q(\omega,t)^{\frac{\sigma-1}{\sigma}} d\omega + \int_{\Omega^*(t)} q(\omega,t)^{\frac{\sigma-1}{\sigma}} d\omega\right]^{\frac{\sigma}{\sigma-1}}$$
$$\int_{\Omega(t)} p(\omega,t)q(\omega,t)d\omega + (1+\tau(t)) \int_{\Omega^*(t)} p(\omega,t)q(\omega,t)d\omega = 1 + \int_{\Omega(t)} \pi(\omega)d\omega + R(t)$$

The last line is the budget constraint. $\pi(\omega, t)$ is profits of a firm ω. $R(t)$ is tax revenue at time t. When trade costs are iceberg costs, this term is equal to zero. Let the right hand side be equal to $I(t)$ (for income). The demand of a particular good is

$$q(\omega,t) = \begin{cases} p(\omega,t)^{-\sigma} P(t)^{\sigma-1} I(t) \text{ if } \omega \in \Omega(t) \\ ((1+\tau(t))p(\omega,t))^{-\sigma} P(t)^{\sigma-1} I(t) \text{ if } \omega \in \Omega^*(t) \\ 0 \text{ otherwise} \end{cases} \quad (2)$$

where $P(t)$ is the Dixit-Stiglitz aggregate price,

$$P(t) = \left[\int_{\Omega(t)} p(\omega,t)^{1-\sigma} d\omega + (1+\tau)^{1-\sigma} \int_{\Omega^*(t)} p(\omega,t)^{1-\sigma} d\omega\right]^{\frac{1}{1-\sigma}} \quad (3)$$

Firms take the demand function 2 as given to determine the prices and quantities given their productivity. This is a static maximization problem, and its solution is to set price equal to a constant mark-up over marginal cost. That is,

$$p(\omega,t) = \frac{\sigma}{\sigma-1} z(\omega,t)^{\sigma-1}$$

where $z(\omega, t)$ is the productivity z of the firm producing good ω at time t. Let $\pi_d(P, I, z)$ be the variable profits for a non exporter (profits before paying innovation or exporting costs). It is straightforward to show that profits for non exporters before paying for innovation expenses are

$$\Pi_d(z, P, I) = \sigma^{-1} I P^{\sigma-1} z^{\sigma-1} = \pi_d(z, P, I) z^{\sigma-1} \tag{4}$$

Variable profits for exporters depend on whether trade costs are tariffs or iceberg transport costs. With tariffs,

$$\Pi_x(z, P, I, \tau) = (1 + (1+\tau)^{-\sigma}) \pi_d(z, P, I) z^{\sigma-1} = \pi_x(z, P, I, \tau) z^{\sigma-1} \tag{5}$$

With iceberg transport costs,

$$\Pi_x(z, P, I, \tau) = (1 + (1+\tau)^{1-\sigma}) \pi_d(z, P, I) z^{\sigma-1} = \pi_x(z, P, I, \tau) z^{\sigma-1} \tag{6}$$

From this point onwards, as is common in Dixit and Stiglitz (1977) models, it is convenient to change variables from the ω state to the z state, since firm decisions depend on the productivity of a firm and not on the name of the good. Let $\mu(z, t)$ be the measure of firms with productivity z at time t. Abusing our notation, the price of a good with productivity z is $p(z, t)$.

Firms decide how much to innovate each period, and non exporters choose whether to become exporters. We start by solving the problem of exporters. Their Hamilton-Jacobi-Bellman equation is

$$(\rho + \delta) V_x(z, \pi_x(t)) = \max_{\dot{z}} \pi_x(t) z^{\sigma-1} - c(z, \dot{z}) + V_{x1}(z, \pi_x(t)) \dot{z} + V_{x2}(z, \pi_x(t)) \dot{\pi}_x(t) \tag{7}$$

For non exporters, the dynamic problem consists on when to become exporters and how much to innovate[1]. Their problem is a stopping time problem:

$$\begin{aligned} V_d(z, \pi_d(t), \pi_x(t)) = \max_{\dot{z}(t), T} \int_0^T & e^{-(\rho+\delta)t} \left[\pi_d(t) z(t)^{\sigma-1} - c(z(t), \dot{z}(t)) \right] dt + \\ & e^{-(\rho+\delta)T} \left[V_x(z(T), \pi_x(T)) - \kappa_X \right] \end{aligned} \tag{8}$$

Let the decision to become an exporter be represented by

$$X(z, \pi_d(t), \pi_x(t)) = \begin{cases} 1 \text{ if become exporter} \\ 0 \text{ if not} \end{cases}$$

[1] We show in the Appendix that a non exporter will always choose to become an exporter.

New firms enter the economy whenever their expected profits exceed the entry cost. That is, in equilibrium, the free entry condition is

$$\kappa_E = \int_1^\infty V_d(z, \pi_d(t), \pi_x(t))(1 - X(z, \pi_d(t), \pi_x(t)))f(z)dz + \int_1^\infty (V_x(z, \pi_x(t))X(z, \pi_d(t), \pi_x(t)) - \kappa_X)f(z)dz \tag{9}$$

3.1 Equilibrium Definition

An equilibrium for this economy is a list of allocations and prices such that

- The consumer demand function is as in (2)
- Firms solve problems (11) and (13) given (4) and (5) if trade costs are tariffs, or (6) if they are iceberg transport costs
- Equation (3) defines the aggregate price index
- The labor market clears: equation (1) holds
- The free entry condition (9) holds
- The evolution of the distribution of firms is consistent with firm innovation decisions.

4 Characterizing the Symmetric Steady State

We focus the characterization of our equilibirum on steady state. The reason is that one purpose of this paper is to characterize the equilibrium in closed form, which we can do for the steady state. In steady state, the aggregate state variables μ, P, and I do not change, so we omit the time index.

The exporter value function is

$$(\rho + \delta)V_x(z) = \max_{\dot{z}} \pi_x z^{\sigma-1} - c(z, \dot{z}) + V_{x1}(z)\dot{z}$$

We solve this problem in the appendix. The solution is the productivity of exporters grows at a constant rate, and is therefore independent of firm size. Thus, Gibrat's law holds. This rate of growth is

$$g_x = \frac{\dot{z}}{z} = \frac{\rho+\delta}{\sigma-1}\left(1-\sqrt{1-h_x}\right)$$
$$h_x = \frac{2\pi_x(\sigma-1)^2}{(\rho+\delta)^2\kappa_1} \tag{10}$$

The rate of growth is increasing in exporter profits and decreasing in innovation costs.

The closed form solution for this value function is

$$V_x(z) = \kappa_1\frac{\rho+\delta}{(\sigma-1)^2}\left(1-\sqrt{1-h_x}\right)z^{\sigma-1} \tag{11}$$

The non exporter value function is

$$V_d(z) = \max_{\dot{z}(t),T}\int_0^T e^{-(\rho+\delta)t}\left[\pi_d z(t)^{\sigma-1} - c(z(t),\dot{z}(t))\right]dt + e^{-(\rho+\delta)T}\left[V_x(z(T)) - \kappa_X\right]$$

This is a stopping time problem. Divide this problem into two steps. Taking T as given, solve the problem

$$\max_{\dot{z}(t)}\int_0^T e^{-(\rho+\delta)t}\left[\pi_d z(t)^{\sigma-1} - c(z(t),\dot{z}(t))\right]dt$$
$$s.t.$$
$$z(0) = \underline{z},\ z(T) = z_x(\underline{z})$$

where $\underline{z}$ is the starting point and $z_x(\underline{z})$ is an ending point, taken as given for now. This is a problem of calculus of variations. We solve it in the Appendix. The solution is that non exporter productivity grows at a constant rate

$$g_d = \frac{\dot{z}}{z} = \frac{\rho+\delta}{\sigma-1}\left(1-\sqrt{1-h_d}\right)$$
$$h_d = \frac{2\pi_d(\sigma-1)^2}{(\rho+\delta)^2\kappa_1} \tag{12}$$

Once again, Gibrat's law holds, and the rate of growth is increasing in non exporter profits, and decreasing in κ_I.

Step 2 requires us to plug in this solution into the value function, and take derivatives with respect to T to find the optimal stopping time. The solution, solved in the Appendix, is that all non exporters grow until they hit a productivity level z_x, point at which they become exporters. Thus, only large firms export.

The non exporter value function is

$$V_d(z) = \frac{\kappa_I g_d}{\sigma - 1} z^{\sigma-1} + \frac{\kappa_x g_d(\sigma-1)}{(\rho+\delta-g_d(\sigma-1))}\left(\frac{z}{z_x}\right)^{\frac{\rho+\delta}{g_d}} \tag{13}$$

And

$$z_x = \left[\frac{(\rho+\delta)\kappa_X(\sigma-1)}{\kappa_I(\rho+\delta-g_d(\sigma-1))(g_x-g_d)}\right]^{\frac{1}{\sigma-1}} \tag{14}$$

Proposition 1 *The smooth pasting condition holds. That is, the non exporter value function smooth pastes into the exporter value function net of the export cost.*

Proof: Rewrite the value functions as

$$\begin{aligned} V_x(z) &= Bz^{\sigma-1} \\ V_d(z) &= Cz^{\sigma-1} + (B-C)z_x^{\sigma-1-\frac{\rho}{g_d}} z^{\frac{\rho}{g_d}} - \kappa_x z_x^{-\frac{\rho}{g_d}} z^{\frac{\rho}{g_d}} \end{aligned}$$

where $B = \kappa_I \frac{\rho}{(\sigma-1)^2}\left(1-\sqrt{1-h_x}\right)$ and $C = \frac{\kappa_I g_d}{\sigma-1}$.

These functions smooth paste in the point z_x. To see this, first notice that $V_d(z_x) = V_x(z_x) - \kappa_x$:

$$V_d(z_x) = Cz_x^{\sigma-1} + (B-C)z_x^{\sigma-1-\frac{\rho}{g_d}} z_x^{\frac{\rho}{g_d}} - \kappa_x z_x^{-\frac{\rho}{g_d}} z_x^{\frac{\rho}{g_d}} = Bz_x^{\sigma-1} - \kappa_x = V_x(z_x)$$

For $z < z_x, V_d(z) > V_d(z) - \kappa_X$. $V_d(z)$ is not defined for $z > z_x$. □

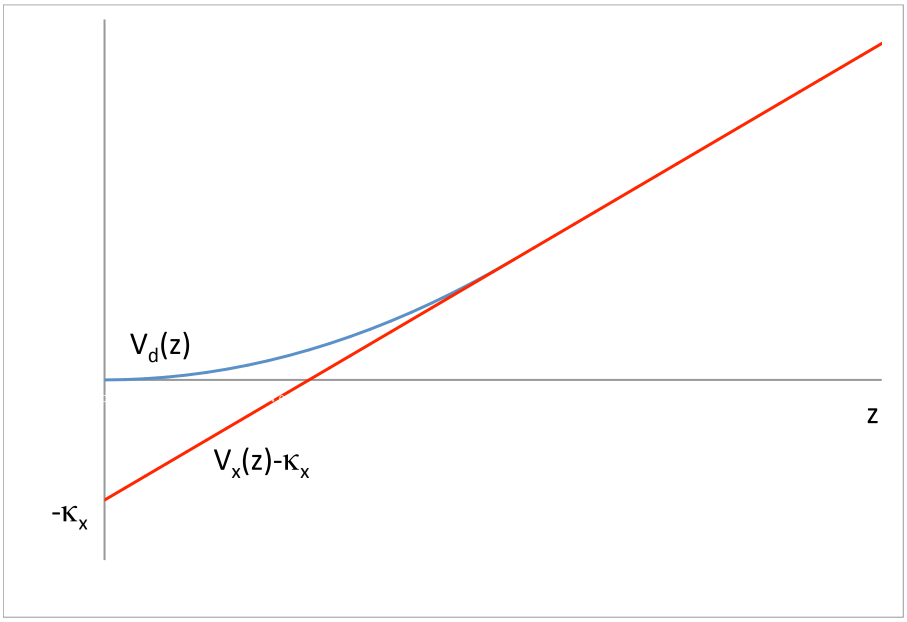
Figure 3 shows these value functions for $\sigma = 2$, in which case $V_x(z)$ is linear.

We next characterize the steady state distribution. Define $\hat{\mu}(t,z)$ as the measure of firms with productivity z in period t. Define $\mathcal{Z} = [z_1, z_2]$. The following expression is the law of motion for the measure of productivity:

$$\hat{\mu}(t+dt,\mathcal{Z}) = \int_{\mathcal{Z}} \hat{\mu}(t, z-\dot{z}dt)e^{-\delta dt}dz + \int_{\mathcal{Z}}\int_0^{dt} M(t)f(z-\dot{z}s)e^{-\delta s}dsdz$$

That is, the measure of firms with productivity $z \in \mathcal{Z}$ is the sum of the incumbent firms that had a productivity $z - \dot{z}dt, dt$ periods ago, plus all the firms that were born and in period $t+dt$ had productivity $z \in \mathcal{Z}$. In the Appendix, we show that this expression can

Figure 3: The Smooth Pasting Condition for $\sigma = 2$



be reduced to

$$\mu(z) = Mf(z)dt + e^{-\delta dt}\mu(z - dz)$$

where $\mu(z) = \hat{\mu}(t, z)$ in steady state.

For dt small enough

$$e^{-\delta dt} \approx 1 - \delta dt$$
$$\mu(z - dz) \approx \mu(z) - \mu'(z)dz \approx \mu(z) - \mu'(z)\dot{z}dt$$

Thus,

$$\mu(z) = Mf(z)dt + (1 - \delta dt)(\mu(z) - \mu'(z)\dot{z}dt)$$
$$\mu(z) = Mf(z)dt + \mu(z) - \delta dt\mu(z) - \mu'(z)\dot{z}dt + \mu'(z)\dot{z}\delta dt^2$$

Ignoring terms with dt^2 and canceling

$$\delta\mu(z)dt = Mf(z)dt - \mu'(z)\dot{z}dt \tag{15}$$

Divide by dt,

$$\delta\mu(z) = Mf(z) - \mu'(z)\dot{z}$$

For non exporters, that is, $z \in [1, z_x]$, this is

$$\delta\mu(z) = Mf(z) - \mu'(z)g_d z$$

This is a first order differential equation, with boundary condition $\mu(1) = Mf(1)$. The solution to this equation is

$$\mu(z) = z^{-\frac{\delta}{g_d}} \left[G_d(z) + Mf(1) - G_d(1)\right]$$

where $G_d(z) = \frac{M}{g_d} \int z^{\frac{\delta}{g_d}-1} f(z)dz$. Similarly, for exporters, that is, for $z > z_x$,

$$\delta\mu(z) = Mf(z) - \mu'(z)g_x z$$

The boundary condition for this differential equation is $\mu(z_x) = z_x^{-\frac{\delta}{\Delta n}} \left[G_d(z) + Mf(1) - G_d(1)\right]$.

The solution is

$$\mu(z) = z^{-\frac{\delta}{g_x}} \left[G_x(z) + z_x^{\frac{\delta}{g_x}-\frac{\delta}{g_d}} \left[G_d(z_x) + Mf(1) - G_d(1)\right] - G_x(z_x)\right]$$

Gathering all together

$$\mu(z) = \begin{cases} z^{-\frac{\delta}{g_d}} \left[G_d(z) + Mf(1) - G_d(1)\right] \text{ if } z \leq z_x \\ z^{-\frac{\delta}{g_x}} \left[G_x(z) + z_x^{\frac{\delta}{g_x}-\frac{\delta}{g_d}} \left[G_d(z_x) + Mf(1) - G_d(1)\right] - G_x(z_x)\right] \text{ if } z > z_x \end{cases} \quad (16)$$

where $G_x(z) = \frac{M}{g_x} \int z^{\frac{\delta}{g_x}-1} f(z)dz$.

Proposition 2 *For a wide family of distributions $f(z)$, $\mu(z)$ features Zipf's law. That is, as z grows, $\mu(z)$ approaches a Pareto distribution.*

Proof: We next show that the upper tail of this distribution satisfies Zipf's law. For this, focus on the segment $z > z_x$, which is the upper tail. Notice that

$$\mu(z) = z^{-\frac{\delta}{g_x}} M \int z^{-\frac{\delta}{g_x}-1} f(z) + KMz^{-\frac{\delta}{g_x}} \text{ for } z > z_x$$

where K is a constant. Taking limits as z grows of the first term in the right hand side,

$$\lim_{z\to\infty} \frac{\frac{\delta}{g_x} M \int z^{-\frac{\delta}{g_x}-1} f(z)dz}{z^{-\frac{\delta}{g_x}}} = \lim_{z\to\infty} \frac{\frac{\delta}{g_x} M z^{-\frac{\delta}{g_x}-1} f(z)dz}{\frac{\delta}{g_x} z^{\frac{\delta}{g_x}-1}} = \lim_{z\to\infty} Mf(z)$$

Thus,

$$\lim_{z\to\infty} \mu(z) = \lim_{z\to\infty} M\left(f(z) + Kz^{-\frac{\delta}{g_x}}\right)$$

If $f(z)$ goes to zero sufficiently fast (faster than $z^{-\delta/g_x}$), then in the limit, $\mu(z)$ approaches a Pareto distribution, that is, Zipf's law holds. Examples of $f(z)$ that go fast enough to zero are the uniform distribution, the normal distribution, and the Pareto as long as the curvature parameter is larger than δ/g_x. □

We can solve for the entire steady state equilibrium as a system of three equations and three unknowns. The unknowns are P, I, M. Given these variables, we can identify all the remaining variables in the model. The three equations that pin down these variables are the index price equation, the free entry condition, and labor market clearing. These equations are:

$$\begin{aligned}
P^{1-\sigma} &= \int_1^{z_x} p(z)^{1-\sigma}\mu(z)dz + (1+\tau)^{1-\sigma}\int_{z_x}^{\infty} p(z)^{1-\sigma}\mu(z)dz \\
\kappa_E &= \int_1^{z_x} V_d(z)f(z)dz + \int_{z_x}^{\infty} \left[V_x(z) - \kappa_x\right] f(z)dz \\
1 &= \int_1^{z_x} \left[\pi_d z^{\sigma-1} + \frac{\kappa_I z^{\sigma-1}}{2} g_d^2\right]\mu(z)dz + \int_{z_x}^{\infty} \left[\pi_x z^{\sigma-1} + \frac{\kappa_I z^{\sigma-1}}{2} g_x^2\right]\mu(z)dz
\end{aligned}$$

where

$$\begin{aligned}
p(z) =& \frac{\sigma}{\sigma-1} z^{\sigma-1} \\
\pi_d =& IP^{\sigma-1} \\
\pi_x =& \begin{cases} (1+(1+\tau)^{-\sigma})\pi_d \text{ if trade costs are tariffs} \\ (1+(1+\tau)^{1-\sigma})\pi_d \text{ if trade costs are iceberg costs} \end{cases}
\end{aligned}$$

$V_d(z), V_x(z), g_d, g_x$ and z_x are as in equations (13), (11), (12), (10) and (14), respectively.

5 Characterizing the Transitional Dynamics

We compute the equilibrium during the transition between steady states by solving a system of partial differential equations. These partial differential equations (PDEs) are given by the measure of exporters and non exporters at each point in time. We assume the economy is in the high trade cost steady state for all $t < 0$, and at $t = 0$ there is an unexpected (small) reduction in trade costs. Trade costs remain at this low value for all $t \geq 0$.

We make the simplifying assumption that new firms enter the economy with common productivity $z = 1$. This implies that the free entry condition is[2]

$$\begin{aligned}\kappa_E =& V_d(1) = \\ =& \int_0^{z_x} \left(\Pi_d(t) - \frac{\kappa_1}{2} g_d(t)\right) e^{(g_d(t)-\rho-\delta)(\sigma-1)t} dt + \int_{z_x}^{\infty} \left(\Pi_x(t) - \frac{\kappa_1}{2} g_x(t)\right) e^{(g_x(t)-\rho-\delta)(\sigma-1)t} dt\end{aligned}$$

This equation is key to our solution strategy. Suppose that $Pi_d(t) = Pi_{d1}$ for all $t \geq 0$. This would imply that $Pi_x(t) = Pi_{x1}, g_d(t) = g_{d1}, g_x(t) = g_{x1}$ for all $t \geq 0$. If this is the case, then the free entry condition is one equation and one unknown for all $t \geq 0$, where the unknown is Π_{d1}. But it turns out that we know the solution to this equation. The equilibrium Π_{d1} is the new steady state level of profits, since the same free entry condition must be satisfied in the steady state with low trade costs.

If these levels of profits and growth rates satisfy the entire system of equations that characterize the equilibrium, then we have found an equilibrium transition path. We show in the Appendix that these equations are satisfied in equilibrium.

This simplifies the analysis a great deal. It implies that we can take profits and growth rates as given to solve for the PDEs that characterize the transition. We solve these PDEs in the Appendix . Here, we lay out our results, that is, the measure of firms along the transition as well as aggregate variables such as price indices and consumption levels.

Our solution strategy divides time into intervals of length t_1, where t_1 is the time it takes a new born firm to become an exporter, that is, $e^{t_1 g_{d1}} = z_{x1}$. The reason for this is that we know that for all $t > t_1$, all firms born before the reduction in trade costs are exporting. This helps us solve the system of equations by grouping cohorts of firms according to their export status. For example, for $t \in (0, t_1)$, firms born before the reduction in trade costs are both exporters and non exporters, but for $t > t_1$, these are all exporters.

[2]Since the drop in trade costs is small enough, this condition holds at every point in time.

Let $\mu(t,z)$ denote the measure of firms with productivity z at time t. Essentially, the PDE to solve is

$$\mu_t(t,z) + \mu_z(t,z) g_i z = -\delta \mu(t,z) \tag{17}$$

where $i = x1$ if the firm is an exporter, and $i = d1$ if it is not. Notice that this equation is the same as the steady state equation where $\mu_t(t,z) = 0$. Equation 17 is the Kolmogorov forward equation along the transition. This can be solved given the value of g_i and an initial condition.

Thus, to solve, we need to take into account that different firms differ in their initial condition. This implies identifying different areas at different points in time and treat them separately. A full description of the procedure is in the Appendix.

We next describe the solution to these PDEs for two time intervals: $t \in (0, t_1)$ and $t \in (t_1, 2t_1)$:

For $t \in (0, t_1)$,

$$\begin{aligned}
\mu(t,z) &= M(t - \frac{1}{g_{1,d}} \log(z)) z^{-\frac{\delta}{g_{1,d}}} && \text{if } z \leq z^*(t) \\
&= M_0 e^{t\delta \frac{g_{1,d} - g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^*(t) < z \leq z_{1,x} \\
&= M_0 z_{1,x}^{\frac{\delta}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}} - 1\right)} e^{t\delta\left(\frac{g_{1,d}}{g_{0,d}} - 1\right)} z^{-\delta \frac{g_{1,d}}{g_{1,x} g_{0,d}}} && \text{if } z_{1,x} < z \leq z^{1*}(t) \\
&= M_0 e^{t\delta \frac{g_{1,x} - g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^{1*}(t) < z \leq z^{3*}(t) \\
&= M_0 e^{t\delta \frac{g_{1,x} - g_{0,x}}{g_{0,x}}} z^{-\frac{\delta}{g_{0,x}}} && \text{if } z > z^{3*}(t)
\end{aligned}$$

where $z^*(t) = e^{g_{1,d} t}$, $z^{1*}(t) = e^{g_{1,x} t} z_{1,x}$, and $z^{3*}(t) = e^{g_{1,x} t} z_{0,x}$. A subsript 0 indicates old steady state. A subscript 1 indicates the new steady state.

For $t_1 < t < 2t_1$

$$\begin{aligned}
\mu(t,z) &= M\left(t - \frac{\log(z)}{g_{1d}}\right) z^{-\frac{\delta}{g_{1d}}} \qquad \text{if } z \le z_{1x} \\
&= M\left(t - \frac{\log(z)}{g_{1x}} + \left(\frac{1}{g_{1x}} - \frac{1}{g_{1d}}\right)\log(z_{1x})\right) z_{1x}^{\delta\left(\frac{1}{g_{1x}} - \frac{1}{g_{1d}}\right)} z^{-\frac{\delta}{g_{1x}}} \qquad \text{if } z_{1x} \le z \le z^{2*}(t) \\
&= M_0 z_{1x}^{\frac{\delta}{g_{0d}}\left(\frac{g_{1d}-g_{1x}}{g_{1x}}\right)} e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z^{-\delta\frac{g_{1d}}{g_{1x}g_{0d}}} \qquad \text{if } z^{2*}(t) \le z \le z^{1*}(t) \\
&= M_0 e^{t\delta\left(\frac{g_{1x}-g_{0d}}{g_{0d}}\right)} z^{-\frac{\delta}{g_{0d}}} \qquad \text{if } z^{1*}(t) \le z \le z^{3*}(t) \\
&= M_0 e^{t\delta\left(\frac{g_{1x}-g_{0x}}{g_{0x}}\right)} z^{-\frac{\delta}{g_{0x}}} \qquad \text{if } z > z^{3*}(t)
\end{aligned}$$

where $z^{2*}(t) = z_{1x}e^{g_{1x}(t)}$.

Notice that to know the measure of firms, we still need to determine entry at each point, that is, $M(t)$ for $t > 0$. We do this via the labor market clearing condition. The solution implies that entry takes the following shape

For $t \in (0, t_1)$,

$$M(t) = m_0 + m_1 e^{m_2 t} + m_3 e^{m_4}$$

And for $t \in (t_1, 2t_1)$,

$$M(t) = \tilde{m}_0 + \tilde{m}_1 e^{\tilde{m}_2 t} + \tilde{m}_3 e^{\tilde{m}_4}$$

where the $m's$ and $\tilde{m}'s$ are constants. See the appendix for details.

6 Lowering Trade Costs

This section describes the exercise we carry out to study the effects of lower trade costs. We start with a simplified version of our model, in which the measure of exporters is set exogenously. This corresponds to cases in which Atkeson and Burstein (2010) have found that innovation does not contribute to welfare gains when the interest rate is zero or when every firm exports. We show that this result holds in our model as well.

We next explore the effects of a reduction in trade costs in the full fledge version of our model. To describe the way we decompose the different sources of productivity and welfare gains. This decomposition depends on whether trade costs are iceberg or tariffs.

6.1 Inelastic Exporting

Before turning to analyze the effect of trade costs using our model, we study the effects on a reduced version of our model, a model in which the measure of exporters is set exogenously, as in the analytical section in Atkeson and Burstein (2010). We prove results analytically assuming that the proportion of firms that export is fixed and exogenous. A proportion λ of firms face an infinite export cost, and therefore choose not to export independently of the variable trade costs, while the remaining firms face zero export costs and always export.

The derivation of our results focuses on the resources devoted to production. We show that when the interest rate is zero, the proportion of resources devoted to production is independent of trade costs. This implies that the aggregate variable profits do not change, which in turn implies that aggregate innovation does not change (where innovation is measured now as the sum of innovation as previously defined, plus resources devoted to firm entry and entry into the export market).

In addition, when $\lambda = 0$ (every firm exports), a drop in trade costs does not increase aggregate innovation even with positive interest rates. The reason for this is that when all firms export, and in particular, entrants export, a reduction in trade costs makes it more profitable to export, and therefore firms would like to increase innovation. But if all firms demand more labor for innovation, and the amount of labor is fixed, the relative price of innovation must increase enough to drive demand for innovation to its previous level.

When $\lambda \in (0, 1)$, we find that a drop in trade costs does not increase aggregate innovation only when the interest rate is zero. Exporters increase innovation, but non exporters reduce their innovation, exactly offsetting the increase by exporters. This result no longer holds under a positive interest rate.

These results are the same as in Atkeson and Burstein (2010), which shows that our model results are consistent with theirs.

Proposition 3 *If the proportion of exporters is independent of variable trade costs and the interest rate is zero, a drop in trade costs does not affect aggregate innovation.*

Proof: The proof shows that the ratio of production workers to workers in the innovation sector does not depend on trade costs. Given that the total number of workers is fixed, the fact that these shares remain constant implies that aggregate innovation does not depend on trade costs.

First note that production workers are proportional to variable profits (before paying for innovation or sunk export costs). For $i = x, d$,

$$\pi_i(z) = \frac{p(z)q_i(z)}{\sigma} = p(z)q_i(z) - h_i(z) \Rightarrow$$
$$h_i(z) = \pi_i(z)\left(\frac{\sigma - 1}{\sigma}\right)$$

This result implies that we can focus on variable profits as opposed to production workers.

The average variable profits for an exporting firm is

$$\bar{\pi}_x = \int_1^\infty \int_0^\infty \pi_x z^{\sigma-1} e^{(g_x(\sigma-1)-\delta)t} f(z) dt dz \Rightarrow$$
$$\bar{\pi}_x = E_f(z^{\sigma-1}) \frac{\pi_x}{\delta - g_x(\sigma - 1)} \tag{18}$$

Where E_f is the expectation operator under the distribution f. To see this, notice that an exporter born with productivity $\bar{z}$ with age t makes profits $e^{g_x t}\pi_x \bar{z}^{\sigma-1}$. The total number of exporters born with productivity $\bar{z}$ is $(1-\lambda)\int_0^\infty e^{-\delta t} f(\bar{z}) dt$. Thus, the sum of profits of exporters born with productivity $\bar{z}$ is $(1-\lambda)\pi_x \int_0^\infty e^{g_x t} z^{\sigma-1} f(z) dt$. Taking an average across z shows equation (18).

For a non exporting firms, this is

$$\bar{\pi}_d = E_f(z^{\sigma-1}) \frac{\pi_d}{\delta - g_d(\sigma - 1)}$$

Total average variable profits are a weighted sum of these numbers:

$$\bar{\pi} = E_f(z^{\sigma-1}) \left[\lambda \frac{\pi_d}{\delta - g_d(\sigma - 1)} + (1-\lambda) \frac{\pi_x}{\delta - g_x(\sigma - 1)} \right]$$

Similarly, average expenditure on innovation is

$$\bar{E} = E_f(z^{\sigma-1}) \kappa_I / 2 \left[\lambda \frac{g_d^2}{\delta - g_d(\sigma - 1)} + (1-\lambda) \frac{g_x^2}{\delta - g_x(\sigma - 1)} \right]$$

The free entry condition is

$$\kappa_E = \lambda \int_1^\infty V_d(z) f(z) dz + (1-\lambda) \int_1^\infty V_x(z) f(z) dz$$

In this case,

$$V_x(z) = \int_0^{\infty} e^{-(\rho+\delta)t}(e^{g_x t}z)^{\sigma-1}\left(\pi_x - \frac{\kappa_I}{2}g_x^2\right)dt = z^{\sigma-1}\left(\frac{\pi_x - \frac{\kappa_I}{2}g_x^2}{\rho+\delta-g_x(\sigma-1)}\right)$$

$$V_d(z) = \int_0^{\infty} e^{-(\rho+\delta)t}(e^{g_d t}z)^{\sigma-1}\left(\pi_d - \frac{\kappa_I}{2}g_d^2\right)dt = z^{\sigma-1}\left(\frac{\pi_d - \frac{\kappa_I}{2}g_d^2}{\rho+\delta-g_d(\sigma-1)}\right)$$

Then free entry implies

$$\kappa_e + E(z^{\sigma-1})\left[\lambda\frac{\kappa_I g_d^2}{\rho+\delta-g_d(\sigma-1)} + (1-\lambda)\frac{\kappa_I g_x^2}{\rho+\delta-g_x(\sigma-1)}\right] = E(z^{\sigma-1})\left[\lambda\frac{\Pi_d}{\rho+\delta-g_d(\sigma-1)} + (1-\lambda)\frac{\Pi_x}{\rho+\delta-g_x(\sigma-1)}\right]$$

It is straightforward to see that if $\rho = 0$, then $\bar{\pi} = \bar{E}$, and transport costs cannot affect the proportion of innovation workers to production workers. □

Proposition 4 *If all firms export ($\lambda = 0$) a drop in trade costs does not affect aggregate innovation.*

Proof: We leave the formal proof for the Appendix. Intuitively, if all firms enter as exporters, a reduction in trade costs increases profits across the board, which increases the incentives to enter considerably. More firms wanting to enter implies more demand for labor resources, which is fixed, so the price of labor must increase (or P falls), offsetting the initial increase in profits and leaving the profits (and therefore innovation) unchanged.

6.2 The Full Model

In this section we use our framework to study how a change in trade costs affects welfare. We focus on how the distribution affects these results, especially whether assumptions on entrants are important. The exercise consists in studying how a drop in trade costs affects welfare under different assumptions for the distribution of entrants. Entrants are relatively more abundant in the lower tail of the distribution, so if the lower tail does not matter, changing the distribution of entrants should not affect the welfare gains from trade.

We concentrate on two statistics to study the effects of a drop in trade costs: welfare and productivity. The change in productivity measures the additional output that a production worker can produce with lower trade costs. The change in welfare measure the additional consumption an individual can consume. Since the number of production workers changes when trade costs change, the increase in welfare does not equal the increase in productivity.

Following Atkeson and Burstein (2010), we decompose the change in productivity into a direct effect and an indirect effect. The direct effect represents the gains just because of the removal of a distortion, that is, the export cost. That is, this is the gain that occurs keeping constant the dynamic decisions of firms, including entry, exporting, and innovation decisions. The indirect effect is the gain that happens because of the reallocation of resources in the economy. We further decompose the indirect effect into the gains from a change in entry, from a change in innovation by exporters, and the change in innovation by non exporters.

Let N_p be total labor used for production.

$$N_p = \int_1^{z_x} n(z)\mu(z)d(z) + \int_{z_x}^{\infty} n(z)\mu(z)dz$$

From the production function, $n(z) = q(z)z^{-1}$. From the demand function,

$$q(z) = \begin{cases} p(z)^{-\sigma}P^{\sigma-1}I \text{ if the good is not exported} \\ (1+(1+\tau)^{-\sigma})p(z)^{-\sigma}P^{\sigma-1}I \text{ if the good is exported} \end{cases}$$

The pricing rule sets $p(z) = \frac{\sigma}{\sigma-1}z^{-1}$. From the budget constraint, $PQ = I$. Combining these expressions, production labor in each firm is

$$n(z) = \begin{cases} P^{\sigma}Qz^{\sigma-1} \text{ if the good is not exported} \\ (1+(1+\tau)^{1-\sigma})P^{\sigma}Qz^{\sigma-1} \text{ if the good is exported} \end{cases}$$

Thus,

$$N_p = P^{\sigma}Q\left[\int_1^{z_x} z^{\sigma-1}\mu(z)dz + \int_{z_x}^{\infty} z^{\sigma-1}(1+(1+\tau)^{1-\sigma})\mu(z)dz\right] \tag{19}$$

From the price index equation (3),

$$\begin{aligned}N_p =& Q\left[\int_1^{z_x} z^{\sigma-1}\mu(z)dz + \int_{z_x}^{\infty} z^{\sigma-1}((1+\tau))^{1-\sigma}\mu(z)dz\right]\times \\ &\left(\frac{\sigma}{\sigma-1}\right)^{\sigma}\left[\int_1^{z_x} z^{\sigma-1}\mu(z)dz + \int_{z_x}^{\infty} z^{\sigma-1}((1+\tau))^{1-\sigma}\mu(z)dz\right]^{\frac{\sigma}{1-\sigma}} \\ =& Q\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}\left[\int_1^{z_x} z^{\sigma-1}\mu(z)dz + \int_{z_x}^{\infty} z^{\sigma-1}((1+\tau))^{1-\sigma}\mu(z)dz\right]^{\frac{1}{\sigma-1}}\end{aligned}$$

Let $Z_d = \frac{\int_1^{z_x} z^{\sigma-1}\mu(z)dz}{M}$ and $Z_x = \frac{\int_{z_x}^{\infty} z^{\sigma-1}\mu(z)dz}{M}$. We can interpret Z_d as the average productivity of a non exporter, and Z_x as the average productivity of an exporter. Then

$$Q = \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}\left[M\left(Z_d + (1+(1+\tau)^{1-\sigma})Z_x\right)\right]^{\frac{1}{\sigma-1}} L_p \tag{20}$$

The term $[M(Z_d + (1+(1+\tau)^{1-\sigma})Z_x)]^{\frac{1}{\sigma-1}}$ is productivity[3]. The average productivity per firm is

$$Z = \frac{[M(Z_d + (1+(1+\tau)^{1-\sigma})Z_x)]^{\frac{1}{\sigma-1}}}{M}$$

A change in trade costs changes Z. It is interesting to understand the composition of these changes. The next proposition, decomposes the change in productivity into several components.

Proposition 5 *The total change in productivity from a change in trade costs and be decomposed into a direct effect and an indirect effect. Moreover, the indirect effect can be decomposed into an entry effect, a non exporter productivity effect, and an exporter productivity effect. The following expression describes this decomposition:*

$$\begin{aligned}\Delta\log(Z) =& \underbrace{-s_x\Delta\log(D)}_{\text{direct effect}} \\ &+ \underbrace{\frac{1}{\sigma-1}\left[\underbrace{\Delta\log(M)}_{\text{entry effect}} + \underbrace{\left(1 - s_x\frac{1+D^{1-\sigma}}{D^{1-\sigma}}\right)\Delta\log(Z_d)}_{\text{non exporter productivity effect}} + \underbrace{s_x\frac{1+D^{1-\sigma}}{D^{1-\sigma}}\Delta\log(Z_x)}_{\text{exporter productivity effect}}\right]}_{\text{indirect effect}}\end{aligned} \tag{21}$$

[3]This is not the way the national accounts compute the productivity of a country (see Rubini (2011)).

where

$$
\begin{aligned}
s_x &= \frac{Z_x D^{1-\sigma}}{\tilde{Z}}, \tilde{Z} = Z^{\sigma-1} \\
D &= 1+\tau
\end{aligned}
$$

Proof: Recall that for every $X \in \mathcal{R}$

$$\frac{\Delta X}{X} = \Delta \log(X)$$

Take logs of Z

$$\log Z = \frac{1}{\sigma-1}\left[\log M + \log\left(Z_d + (1+D^{1-\sigma})Z_x\right)\right]$$

Now take derivatives

$$\Delta \log Z = \frac{1}{\sigma-1}\left[\Delta \log M + \Delta \log \tilde{Z}\right]$$

$$\Delta \log \tilde{Z} = \frac{1}{\tilde{Z}}\left(\Delta Z_d + (1+D^{1-\sigma})\Delta Z_x + (1-\sigma)D^{-\sigma}\Delta D Z_x\right)$$

From the definition of s_x,

$$
\begin{aligned}
\frac{D^{-\sigma}\Delta D Z_x}{\tilde{Z}} &= s_x\frac{\Delta D}{D} = s_x \Delta \log D \\
\frac{Z_x}{\tilde{Z}} &= \frac{s_x}{D^{1-\sigma}} \\
(1+D^{1-\sigma})\frac{\Delta Z_x}{\tilde{Z}} &= (1+D^{1-\sigma})\frac{\Delta Z_x}{Z_x}\frac{Z_x}{\tilde{Z}} = s_x\frac{(1+D^{1-\sigma})}{D^{1-\sigma}}\Delta \log Z_x \\
\frac{Z_d}{\tilde{Z}} &= 1 - \frac{(1+D^{1-\sigma})Z_x}{\tilde{Z}} = 1 - s_x\frac{(1+D^{1-\sigma})}{D^{1-\sigma}} \\
\frac{\Delta Z_d}{\tilde{Z}} &= \frac{\Delta Z_d}{Z_d}\frac{Z_d}{\tilde{Z}} = \left(1 - s_x\frac{(1+D^{1-\sigma})}{D^{1-\sigma}}\right)\Delta \log Z_d
\end{aligned}
$$

□

Note that this derivation works well only for infinitesimal changes.

7 Results

In this section we present the quantitative effects of dropping trade costs on productivity and welfare. Furthermore, we decompose the productivity effects into a direct effect and an indirect effect. The indirect effect is further deconstructed into an entry effect, an exporter productivity effect, and a non exporter productivity effect, as specified in equation (21).

We next describe the way we discipline our parameters. Our goal is to bring discipline in similar ways to models in international trade, and then explore the role of changing the density of firms in the lower tail. Two common targets in the literature are: (i) the export volume, as a ratio of total GDP; and (ii) the shape of the distribution of firms among large firms. Thus, all the cases we study are calibrated so that the export to GDP ratio is the same and the slope of the distribution of firms around the upper tail is the same.

We set the initial trade cost $\tau = 0.1$. We set the export volume equal to 10%. The slope of the distribution of firms in the upper tail is determined by the rate of growth of exporting firms. We set this equal to 10%, which implies a slope in the upper tail of -1.5. We also set $\kappa_E = 7$.

Next we need to choose a functional form for the distribution of entrants. This function determines the relative number of small firms, since entrants are relatively small firms. Thus, if entrants tend to be concentrated around one point, then the relative number of small firms is large. We choose a Pareto distribution[4] because it is easier to work with this kind of distribution. Therefore:

$$z \sim f(z) = \theta z^{-\theta-1}, z \in [1, \infty)$$

To change the relative number of small firms we change the parameter θ. A larger θ implies a larger relative number of small firms.

We also experiment with the case when all entrants enter as non exporters with $z = 1$, that is, the lowest possible draw. This corresponds to the case with $\theta = \infty$.

We then drop our trade costs to $\tau = 0.09$ and compute the changes in export volumes, welfare, and productivity. We report our results as elasticities. That is, the numbers we show represent the change per percentage point drop in $(1 + \tau)$.

Table 1 shows our results when trade costs are iceberg transport costs. It shows that

[4]The Pareto shape of the endogenous distribution does not depend on this assumption.

the assumptions we make about small firms matter, and they matter quite a lot. Our welfare calculations of reducing an iceberg transport cost by one percentage point vary from a gan of 0.36 percentage points to as much as 2.67 percentage points! When entrants are relatively small firms (large θ), reducing trade costs has much larger effects on welfare, productivity, and the trade volume. And the increase in productivity takes place via the indirect effect. With relatively small entrants, a change in trade costs produces more reallocation of resources, and a larger aggregate effect.

Table 1: Lowering an Iceberg Transport Cost

θ	2.5	3	4	∞
Export volume	4.20	4.66	5.33	5.97
Productivity	0.17	0.18	0.20	1.30
Direct effect	0.10	0.10	0.10	0.10
Indirect effect	0.07	0.08	0.10	1.20
Entry effect	-0.74	-0.85	-1.01	-0.08
Productivity avg. firm	0.82	0.94	1.11	1.28
Non exporter effect	-0.05	-0.05	-0.06	-0.06
Exporter effect	0.87	0.99	1.09	0.08
Welfare	0.36	0.40	0.45	2.67

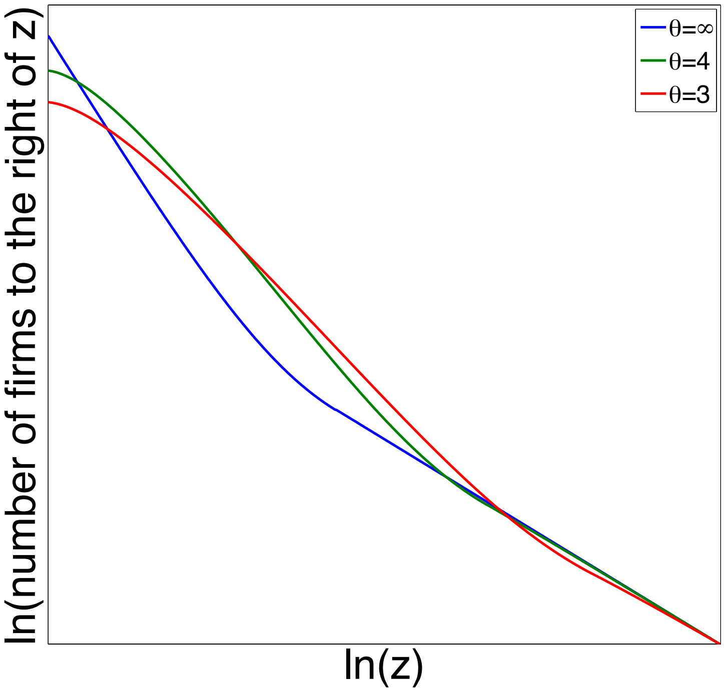
7.1 The Transition

We now present the results of a reduction in trade costs on welfare by considering the transition to the new steady state. We compute the transition as specified in section 5, which implies that entrants enter with $z = 1$. We assume that the economy reaches the news steady state in $2t_1$ periods. This implies that we assume that welfare jumps discretely from its level in $t = 2t_1$ to the steady state level. We do this because these levels are very close. For example, assuming that welfare stays at the level it has in $t = 2t_1$ would produce almost no change.

Table 2 shows the gains welfare. Our welfare metric is the equivalent variation in consumption from a change in marginal trade costs, defined as the change in consumption at the old steady state that leaves households indifferent between the old steady state and the transition to the new steady state.

We report these increases as elasticities and compare them with the effect of reducing trade costs in a model in which every firm exports. In this model, innovation plays no

Figure 4: The Endogenous Distribution of Firms



role, as described in section 6.1. We calibrate the variable trade costs in this model so that the trade volume relative to total output is the same as in the full fledge model.

Table 2: Welfare Gains in the Transition

	Iceberg Costs
Change in Welfare across steady states	2.67
Total Discounted Increase in Welfare	1.00
Change in Welfare in a model where every firm exports	0.10
Gain relative to all firms export	0.90

The present value value of the increase in welfare is 90 percent larger than what a model without innovation would observe in the case of iceberg costs. These numbers are large, and it implies that innovation is a key determinant of welfare gains.

8 Discussion: Comparison to the Literature

As established in the previous section, we find that the indirect effect is an important component of welfare gains from trade. That is, innovation matters. This contrasts the results in Atkeson and Burstein (2010), who find that the transition is so slow, that the discounted flow of welfare is barely larger than a model with no innovation. We focus on the model with iceberg costs, the assumption made in Atkeson and Burstein (2010).

Our estimates of the gains in models with no innovation are very close. We find an elasticity of welfare to tariffs of 0.1 percent, while Atkeson and Burstein (2010) find it is 0.08 percent.

The difference between both results emerge in two different calculations: the comparison of steady states, and the comparison of transitions. Comparing steady states, we find that reducing tariffs by one percent increases welfare across steady states by 267 percent. Atkeson and Burstein (2010) find this number equal to 75 percent in their largest estimate. Thus, we compute gains that are 4 times as large. This is not surprising given that we deal with very different models.

The most surprising difference comes from the transition. We find that the discounted gains are 90 percent larger than the model with no innovation. This is much larger that Atkeson and Burstein (2010)'s estimate of 0.8 percent larger. Thus, the relevant question is therefore why our transition speeds are so different.

A key difference with Atkeson and Burstein is that the entrants in their model are relatively large. Precisely, they are about 40% larger than the average firm in the economy.[5] In our case, the average entrant is about 60% as large as the average firm in the economy, which is more consistent with the data (see for example Dunne et al. (1988) and Geroski (1995)).

At this stage, we are still working on understanding the difference between the estimates.

9 Conclusion

We investigate the importance of the distribution of firms in determining welfare gains from trade liberalization. We find that the assumptions we make about the distribution of firms matter when thinking about trade policy. While previous research suggests that the distribution of large firms is important to determine the consequences of a change in trade policy, we show that our assumptions on smaller firms also matter. In particular, the larger the relative measure of small firms, the larger the effects of trade liberalization on welfare and trade volumes.

The finding that small firms are important contradicts existing research such as Gabaix (2011), who suggests that large firms are what really determine aggregate outcomes. This

[5] While this number is not in the paper, we have calculated it using their Matlab codes.

conclusion is based on the fact that the distribution of firms is fat tailed, and the shocks to firms are exogenous.

We relax the assumption of exogenous shocks and study the reaction of firms to a change in trade costs, allowing firms to determine endogenously their size via innovation. We find that while big firms react to lower trade costs by expanding, small firms also react by contracting. In the aggregate, the reaction of small firms is as important as the reaction of large firms. This suggests that all analysis on trade policy should model carefully the entire distribution of firms, not only the large ones.

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A Deriving the Endogenous Distribution of Firms

Define $\mathcal{Z} = [z_1, z_2]$

$$\hat{\mu}(t + dt, \mathcal{Z}) = \int_{\mathcal{Z}} \hat{\mu}(t, z - \dot{z}dt)e^{-\delta dt}dz + \int_{\mathcal{Z}} \int_{t}^{t+dt} M(s)f(z - \dot{z}s)e^{-\delta s}dsdz$$

Taking limits as $z_1 \to z_2 \to z$

$$\hat{\mu}(t + dt, z) = \hat{\mu}(t, z - \dot{z}dt)e^{-\delta dt} + \int_{t}^{t+dt} M(s)f(z - \dot{z}s)e^{-\delta s}ds$$

For small dt, the following holds:

$$\begin{aligned}
\hat{\mu}(t + dt, z) &\approx \hat{\mu}(t, z) + \hat{\mu}_1(t, z)dt \\
\hat{\mu}(t, z - \dot{z}dt) &\approx \hat{\mu}(t, z) - \hat{\mu}_2(t, z)\dot{z}dt \\
f(z - \dot{z}s) &\approx f(z) - f'(z)\dot{z}s \\
e^{-\delta dt} &\approx (1 - \delta dt)
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{\mu}(t, z) + \hat{\mu}_1(t, z)dt =& \hat{\mu}(t, z) - \hat{\mu}_2(t, z)\dot{z}dt - \delta dt\left(\hat{\mu}(t, z) + \hat{\mu}_2(t, z)\dot{z}dt\right) + \\
& \int_{t}^{t+dt} M(s)(f(z) - f'(z)\dot{z}s)(1 - \delta s)ds
\end{aligned}$$

Note that in steady state $\hat{\mu}_1(t, z) = 0$ and $M(s) = M$. Putting all together,

$$\hat{\mu}(t, z) = \hat{\mu}(t, z) - \hat{\mu}_2(t, z)\dot{z}dt - \delta dt\left(\hat{\mu}(t, z) + \hat{\mu}_2(t, z)\dot{z}dt\right) + M \int_{t}^{t+dt} (f(z) - f'(z)\dot{z}s)(1 - \delta s)ds$$

Solving for the last integral

$$\int_t^{t+dt} f(z)ds = f(z)dt$$
$$\int_t^{t+dt} -f'(z)\dot{z}sds = -f'(z)\dot{z}(dt)^2/2$$
$$\int_t^{t+dt} -\delta sf(z)ds = -\delta f(z)(dt)^2/2$$
$$\int_t^{t+dt} -\delta f'(z)\dot{z}s^2ds = -\delta f'(z)\dot{z}(dt)^3/3$$

Eliminating all the terms with dt elevated to a power larger than 1,

$$\hat{\mu}(t,z) = \hat{\mu}(t,z) - \hat{\mu}_2(t,z)\dot{z}dt - \delta\hat{\mu}(t,z)dt + Mf(z)dt$$

Cancelling terms and dividing by dt,

$$\delta\hat{\mu}(t,z) = Mf(z) - \hat{\mu}_2(t,z)\dot{z}$$

B Proof of Proposition 4

We prove that when all firms export (because $\kappa_X = 0$), the elasticity of welfare with respect to a change in tariffs is the same as this elasticity in a model where firms cannot make innovation or entry decisions. Given that these elasticities are the same, innovation does not contribute to welfare when all firms export.

B.1 Model with no change in innovation or entry decisions

We assume that all firms export. Additionally, following a drop in trade costs, firms are not allowed to change their productivity, exporting decisions, or entry decisions. This is the direct effect of section 6.2.

Notice that in this case the measure of firms is exogenous. We normalize the measure of firms equal to 1 without loss of generality, distributed with an exogenous distribution $f(z)$.

The equilibrium conditions are the price equation

$$\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1} P^{1-\sigma} = \left(1+(1+\tau)^{1-\sigma}\right) \int_1^\infty z^{\sigma-1} f(z) dz$$

And labor market clearing:

$$\bar{L} = \Pi_d \left(1+(1+\tau)^{1-\sigma}\right) \int_1^\infty z^{\sigma-1} f(z) dz + \kappa_e$$

Let $\int_1^\infty z^{\sigma-1} f(z) dz = E_f$, the expected value of $z^{\sigma-1}$ under the distribution $f(z)$, we rewrite the above equations as

$$\left(\frac{\sigma}{\sigma-1}\right)^{\sigma-1} P^{1-\sigma} = \left(1+(1+\tau)^{1-\sigma}\right) E_f$$
$$\bar{L} = \Pi_d(\sigma-1)\left(1+(1+\tau)^{1-\sigma}\right) E_f + \kappa_e$$

We know

$$\Pi_d = P^{\sigma-1} I \sigma^{-\sigma} (\sigma-1)^{\sigma-1}$$

Which implies

$$I = \frac{\Pi_d}{P^{\sigma-1}} \frac{\sigma^\sigma}{(\sigma-1)^{\sigma-1}}$$

Welfare is

$$W = C = \frac{I}{P} = \frac{\Pi_d}{P^\sigma} \frac{\sigma^\sigma}{(\sigma-1)^{\sigma-1}}$$

From market clearing

$$\Pi_d = \frac{(\bar{L}-\kappa_e)}{(\sigma-1)\left(1+(1+\tau)^{1-\sigma}\right) E_f}$$

Thus, welfare is

$$W_I = \frac{\Pi_d}{P^\sigma}\frac{\sigma^\sigma}{(\sigma-1)^{\sigma-1}} = \left[\frac{1}{P}\frac{\sigma}{\sigma-1}\right]^\sigma \frac{(\bar{L}-\kappa_e)}{(1+(1+\tau)^{1-\sigma})\, E_f}$$

Notice that from the price equation

$$\left[\frac{\sigma}{\sigma-1}\frac{1}{P}\right]^\sigma = \left[\left(1+(1+\tau)^{1-\sigma}\right) E_f\right]^{\frac{\sigma}{\sigma-1}}$$

Thus,

$$\begin{aligned} W_I =& (\bar{L}-\kappa_e)\frac{[(1+(1+\tau)^{1-\sigma})\, E_f]^{\frac{\sigma}{\sigma-1}}}{(1+(1+\tau)^{1-\sigma})\, E_f} \\ W_I =& (\bar{L}-\kappa_e)\left[\left(1+(1+\tau)^{1-\sigma}\right) E_f\right]^{\frac{1}{\sigma-1}} \end{aligned}$$

Taking logs,

$$\ln W_I = \frac{1}{\sigma-1}\ln((\bar{L}-\kappa_e)E_f) + \frac{1}{\sigma-1}\ln\left(1+(1+\tau)^{1-\sigma}\right)$$

Taking derivatives

$$\frac{\partial \ln W_I}{\partial(1+\tau)} = -\frac{(1+\tau)^{-\sigma}}{(1+(1+\tau)^{1-\sigma})}$$

Multiplying by $(1+\tau)$

$$\eta_I = -\frac{(1+\tau)^{1-\sigma}}{((1+\tau)^{1-\sigma}+1)}$$

For illustration purposes, setting $\sigma = 2$,

$$\eta_I = -\frac{1}{2+\tau}$$

We next show that allowing for innovation and entry decisions does not change this elasticity when all firms export.

B.2 Model with innovation and entry decision, and all firms export

In this section we show that allowing for innovation and entry decisions, the elasticity of welfare with respect to tariffs is given by equation (**??**). Thus, in a model in which all firms export innovation does not increase the response of welfare to a drop in tariffs.

Note: this proof is incomplete. This proof is for $\sigma = 2$. We are working on the proof for a general σ. Also, we do not include here the proof for the case in which transport costs are iceberg. This will also be included shortly.

The proof proceeds as follows. We first derive expressions assuming that the measure of firms M does not change with tariffs. Then we show that this is true. The proof is incomplete since it is done for the case with $\sigma = 2$. However, it works for general values of σ.

Start with the free entry condition:

$$\kappa_e = \int_1^\infty V_x(z) f(z) dz = B \int_1^\infty z^{\sigma-1} f(z) dz = B E_f(z^{\sigma-1})$$

Where $B = \kappa_1 \frac{\rho}{(\sigma-1)^2} \left(1 - \sqrt{1-h_x}\right)$. Let $\tilde{P} = P \times I$. Given any change in trade costs $d\tau$ it must true that:

$$0 = \left[\frac{\partial B}{\partial \tau} d\tau + \frac{\partial B}{\partial \tilde{P}} d\tilde{P}\right] E_f(z) \tag{22}$$

Thus,

$$\frac{d\tilde{P}}{d\tau} = -\frac{\frac{\partial B}{\partial \tau}}{\frac{\partial B}{\partial \tilde{P}}}$$

Differentiating and doing some algebra

$$\frac{\partial B}{\partial \tilde{P}} = \frac{(1 + (1+\tau)^{-\sigma})}{\rho + \delta - g_x}$$

and

$$\frac{\partial B}{\partial \tau} = \frac{-2\tilde{P}(1+\tau)^{-\sigma-1}}{\rho + \delta - g_x}$$

then

$$\frac{d\tilde{P}}{d\tau} = \frac{2\tilde{P}(1+\tau)^{1-\sigma}}{(1+\tau)^{\sigma} + 1} \tag{23}$$

Set $\sigma = 2$.[6] Given $\tilde{P}$ and M, the following must hold:

$$2P^{-1} = \frac{(2+\tau)}{(1+\tau)} \int_1^{\infty} z\mu(z)dz$$

Let $E_\mu(z) = \int_1^\infty z\mu(z)dz$ and notice that it is a function of both $\tilde{P}$ and τ through g_x. Then, differentiating the above equation, and assuming $dM = 0$, we get:

$$-\frac{2dP}{P^2} = -\frac{d\tau}{(1+\tau)^2}E_\mu + \frac{(2+\tau)}{(1+\tau)}\frac{\partial E_\mu}{\partial g_x}\frac{\partial g_x}{\partial \tau}d\tau + \frac{(2+\tau)}{(1+\tau)}\frac{\partial E_\mu}{\partial g_x}\frac{\partial g_x}{\partial \tilde{P}}d\tilde{P}$$

$$-\frac{2dP}{P^2} = -\frac{d\tau}{(1+\tau)^2}E_\mu + \frac{(2+\tau)}{(1+\tau)}\frac{\partial E_\mu}{\partial g_x}\left[\frac{\partial g_x}{\partial \tau}d\tau + \frac{\partial g_x}{\partial \tilde{P}}d\tilde{P}\right] \tag{24}$$

Since B is an affine function of g_x, equation (22) implies $\frac{\partial g_x}{\partial \tau}d\tau + \frac{\partial g_x}{\partial \tilde{P}}d\tilde{P} = 0$, therefore

$$-\frac{2dP}{P^2} = -\frac{d\tau}{(1+\tau)^2}E_\mu$$

$$\frac{2dP}{P} = P\frac{d\tau}{(1+\tau)^2}E_\mu$$

$$\frac{2dP}{P} = \frac{2}{\frac{(2+\tau)}{(1+\tau)}E_\mu}\frac{d\tau}{(1+\tau)^2}E_\mu$$

$$\frac{dP}{P} = \frac{1}{(2+\tau)}\frac{d\tau}{(1+\tau)}$$

This last equation gives the elasticity of the price to changes in tariffs. Next we work an expression that translate the change in price into a change in welfare. Welfare is $W = C = \frac{I}{P}$.

$$d\tilde{P} = \left[\frac{\partial P}{\partial \tau}I + \frac{\partial I}{\partial \tau}P\right]d\tau \tag{25}$$

$$dW = \left[\frac{\frac{\partial I}{\partial \tau}P - \frac{\partial P}{\partial \tau}I}{P^2}\right]d\tau$$

[6] The proof for a general value of σ will come shortly.

$$\frac{dW}{W} = \left[\frac{\frac{\partial I}{\partial \tau}P - \frac{\partial P}{\partial \tau}I}{\tilde{P}}\right] d\tau$$

From equation (25),

$$\frac{\partial I}{\partial \tau}P = \frac{d\tilde{P}}{d\tau} - \frac{\partial P}{\partial \tau}I$$

Thus

$$\frac{dW}{W} = \left[\frac{\frac{d\tilde{P}}{d\tau} - \frac{\partial P}{\partial \tau}I - \frac{\partial P}{\partial \tau}I}{\tilde{P}}\right] d\tau$$

$$\frac{dW}{W} = \frac{d\tilde{P}}{d\tau}\frac{d\tau}{\tilde{P}} - 2\frac{\partial P}{\partial \tau}\frac{d\tau}{P}$$

Using the change on $\tilde{P}$ from free entry we get

$$\frac{dW}{W} = \frac{2(1+\tau)^{-1}}{(1+\tau)^2+1}d\tau - 2\frac{\partial P}{\partial \tau}\frac{d\tau}{P}$$

In addition, from the elasticity of the price change we get,

$$\frac{\partial P}{\partial \tau}\frac{1}{P} = \frac{1}{(2+\tau)(1+\tau)}$$

Then,

$$\frac{dW}{W} = \frac{2(1+\tau)^{-1}}{(1+\tau)^2+1}d\tau - 2\frac{d\tau}{(2+\tau)(1+\tau)}$$

$$\frac{dW}{W} = \frac{-2\tau}{[(1+\tau)^2+1](2+\tau)}d\tau$$

Thus, the elasticity of welfare to changes is in the tariff is

$$\eta_{w,\tau} = \frac{\frac{dW}{W}}{\frac{d\tau}{(1+\tau)}} = \frac{-2\tau(1+\tau)}{[(1+\tau)^2+1](2+\tau)}$$

This is the same expression as equation (**??**). So, assuming $\frac{\partial M}{\partial \tau} = 0$, we have shown that if all firms export innovation does not contribute to the increase in welfare from a drop in tariffs. Next we show $\frac{\partial M}{\partial \tau} = 0$.

Equation (24) becomes

$$-\frac{2dP}{P^2} = -\frac{d\tau}{(1+\tau)^2}E_\mu + \frac{(2+\tau)}{(1+\tau)}\frac{\partial E_\mu}{\partial g_x}\left[\frac{\partial g_x}{\partial \tau}d\tau + \frac{\partial g_x}{\partial \tilde{P}}d\tilde{P}\right] + \frac{(2+\tau)}{(1+\tau)}\frac{\partial E_\mu}{\partial M}dM$$

The free entry condition, guarantees the middle term is zero, so

$$-\frac{2dP}{P^2} = -\frac{d\tau}{(1+\tau)^2}E_\mu + \frac{(2+\tau)}{(1+\tau)}\frac{\partial E_\mu}{\partial M}dM$$

Since the E_μ is a linear function of M we have

$$\frac{\partial E_\mu}{\partial M} = \frac{E_\mu}{M}$$

Then,

$$\frac{2dP}{P^2} = \frac{d\tau}{(1+\tau)^2}E_\mu - \frac{(2+\tau)}{(1+\tau)}\frac{E_\mu}{M}dM$$

Multiplying by P and substituting,

$$\frac{dP}{P} = \frac{1}{(2+\tau)}\frac{d\tau}{(1+\tau)} - \frac{dM}{M} \tag{26}$$

Notice that the market clearing condition can be written as,

$$\bar{L} = \frac{\tilde{P}}{2}[1+(1+\tau)^{-2}]E_\mu + \frac{\kappa_1}{2}g_x^2 E_\mu + \kappa_e M$$

or

$$\bar{L} = \left[\frac{\tilde{P}}{2}[1+(1+\tau)^{-2}] + \frac{\kappa_1}{2}g_x^2\right]E_\mu + \kappa_e M$$

Let $h(\tau,\tilde{P}) = \frac{\tilde{P}}{2}[1+(1+\tau)^{-2}] + \frac{\kappa_1}{2}g_x^2$, then

$$0 = \left[\frac{\partial E_\mu}{\partial \tilde{P}}d\tilde{P} + \frac{\partial E_\mu}{\partial \tau}d\tau + \frac{\partial E_\mu}{\partial M}dM\right]h(\tau,\tilde{P}) + E_\mu\left[\frac{\partial h}{\partial \tilde{P}}d\tilde{P} + \frac{\partial h}{\partial \tau}d\tau\right] + \kappa_e dM$$

For the same arguments used before, that is, $\tilde{P}$ and τ affect μ only trough g_x, we have that $\frac{\partial E_\mu}{\partial \tilde{P}}d\tilde{P} + \frac{\partial E_\mu}{\partial \tau}d\tau = 0$. Then

$$0 = h(\tau,\tilde{P})\frac{E_\mu}{M}dM + E_\mu\left[\frac{\partial h}{\partial \tilde{P}}d\tilde{P} + \frac{\partial h}{\partial \tau}d\tau\right] + \kappa_e dM$$

or

$$\frac{dM}{M} = -\frac{E_\mu\left[\frac{\partial h}{\partial \tilde{P}}d\tilde{P} + \frac{\partial h}{\partial \tau}d\tau\right]}{h(\tau,\tilde{P})E_\mu + \kappa_e M}$$

From the market clearing condition,

$$\frac{dM}{M} = -\frac{1}{\bar{L}} E_\mu \left[\frac{\partial h}{\partial \tilde{P}} d\tilde{P} + \frac{\partial h}{\partial \tau} d\tau \right] \tag{27}$$

The sign and magnitude of the change on M is given by the total differential of the function $h(\tau, \tilde{P})$. This total differential is

$$\frac{\partial h}{\partial \tilde{P}} = \frac{1 + (1+\tau)^{-2}}{2} + \kappa_1 g_x \frac{\partial g_x}{\partial \tilde{P}}$$

$$\frac{\partial h}{\partial \tau} = -\tilde{P}(1+\tau)^{-3} + \kappa_1 g_x \frac{\partial g_x}{\partial \tau}$$

Then, given that $\frac{\partial g_x}{\partial \tau} d\tau + \frac{\partial g_x}{\partial \tilde{P}} d\tilde{P} = 0$,

$$\frac{\partial h}{\partial \tilde{P}} d\tilde{P} + \frac{\partial h}{\partial \tau} d\tau = \frac{1 + (1+\tau)^{-2}}{2} d\tilde{P} - \tilde{P}(1+\tau)^{-3} d\tau$$

Using equation (23) in the above

$$\frac{\partial h}{\partial \tilde{P}} d\tilde{P} + \frac{\partial h}{\partial \tau} d\tau = \left[\frac{[1 + (1+\tau)^{-2}]}{2} \frac{2(1+\tau)^{-1}}{(1+\tau)^2 + 1} - (1+\tau)^{-3} \right] \tilde{P} d\tau$$

$$\frac{\partial h}{\partial \tilde{P}} d\tilde{P} + \frac{\partial h}{\partial \tau} d\tau = 0$$

Thus,

$$\frac{dM}{M} = 0 \tag{28}$$

That is, M does not depend on trade costs, which shows that the gains from trade when all firms export is independent of whether there is innovation or not.

C The Distribution Along the Transition

In this section we derive the partial differential equations that characterize the measure of firms along the transition between steady states following a drop in trade costs. In period $t = 0$, the economy is in the steady state with high tariffs. That period, we introduce an unexpected drop in tariffs. We model the economy for $t > 0$. We assume that all entrants enter with productivity $z = 1$.

As specified in section 5, we take as given the rates of growth and the profits for exporters and non exporters before and after the tariff drop. Also, we take as given the export thresholds.

We proceed as follows. First, we derive the PDEs assuming that there are no exporters, for illustration purposes. Then we add back the non exporters. Thus, assume that no firm exports in what follows.

Define $\mu(t,z)$ as the measure of firms with productivity z at time t. Let $\mu^n(t,z)$ be the measure of firms born in $t>0$ ($n = new$), and $\mu^o(t,z)$ be the measure of firms born in $t<0$ ($o = old$). Consider new firms first. The partial differential equation to be solved is[7]

$$\mu^n_t(t,z) + \mu^n_z(t,z)g_1 z = -\delta\mu^n(t,z)$$

where a subindex denotes a partial derivative.

The solution for this equation is characterized by the characteristics functions

$$dt = \frac{dz}{g_1 z} = -\frac{d\mu^n}{\mu^n\delta} \tag{29}$$

Solving the equality $dt = \frac{dz}{g_1 z}$ we get

$$t - \frac{\log(z)}{g_1} = C_1$$

Where C_1 is a constant of integration. Solving the equality $dt = -\frac{d\mu^n}{\mu^n\delta}$ we get,

$$t + \frac{\log(\mu^n)}{\delta} = C_2$$

And the general solution is given by an arbitrary function F such that

$$C_2 = F(C_1)$$

or

$$t + \frac{\log(\mu^n)}{\delta} = F(t - \frac{\log(z)}{g_1}) \tag{30}$$

We find F using the boundary condition $\mu^n(t,1) = M(t)$, which implies $\frac{\log(\mu^n(t,1))}{\delta} = \frac{\log(M(t))}{\delta}$. From the above equation,

[7]Notice that we recover the steady state Kolmogorov forward equation when $\mu^n_t = 0$.

$$t + \frac{\log(\mu^n(t,1))}{\delta} = F(t)$$

Therefore,

$$F(x) = \frac{\log(M(x))}{\delta} + x$$

Thus, from (30) and replacing $x = t - \frac{\log(z)}{g_1}$ we get that

$$t + \frac{\log(\mu^n)}{\delta} = \frac{\log(M(t - \frac{\log(z)}{g_1}))}{\delta} + t - \frac{\log(z)}{g_1}$$

$$\log(\mu^n(t,z)) = \log(M(t - \frac{\log(z)}{g_1})) - \delta\frac{\log(z)}{g_1}$$

Which implies

$$\mu^n(t,z) = M(t - \frac{1}{g_1}\log(z))z^{-\frac{\delta}{g_1}}$$

Notice that $\frac{1}{g_1}\log(z)$ is the age of a firm, say s. Thus, $t-s$ determines in which period the firm was born and therefore it determines how many other firms alike, $M(t-s)$ are in the economy. Of course, as long as $z \leq z^*(t)$ we know that $t - s \geq 0$ and that $\mu^n(t,z) = \mu(t,z)$ for all $z \leq z^*(t)$. Thus,

$$\mu(t,z) = M(t - \frac{1}{g_1}\log(z))z^{-\frac{\delta}{g_1}} \qquad \text{if } z < z^*(t)$$

Next, focus on firms born in $t < 0$. We know $\mu^o(t,z) = 0$ for $z < z^*(t)$. For $z \geq z^*(t)$, the PDE is

$$\mu^o_t(t,z) + \mu^o_z(t,z)g_1 z = -\delta\mu^o(t,z)$$

The solution is the same as before, but now we write it in a different way

$$e^t z^{-\frac{1}{g_1}} = C_1$$

and

$$e^t(\mu^o)^{\frac{1}{\delta}} = C_2$$

where g_1 is the growth rate after the drop in trade costs. The solution is given for an arbitrary function F satisfying

$$e^t(\mu^o)^{\frac{1}{\delta}} = F[e^t z^{-\frac{1}{g_1}}]$$

The difference with the case solved before is that now to pin down F we use the initial condition $\mu^o(0,z) = M_0 z^{-\frac{\delta}{g_0}}$, where g_0 is the growth rate before the drop in trade costs, which implies

$$(\mu^o)^{\frac{1}{\delta}} = F[z^{-\frac{1}{g_1}}] = M_0^{\frac{1}{\delta}} z^{-\frac{1}{g_0}}$$

Therefore

$$F(x) = M_0^{\frac{1}{\delta}} x^{\frac{g_1}{g_0}}$$

Using the above function in the solution we get

$$e^t(\mu^o)^{\frac{1}{\delta}} = e^{t\frac{g_1}{g_0}} z^{-\frac{1}{g_0}}$$

or

$$\mu^o(t,z) = M_0 e^{t\delta\frac{g_1-g_0}{g_0}} z^{-\frac{\delta}{g_0}} \tag{31}$$

Notice that this equation seems to diverge, especially if $g_1 > g_0$ and it would do so at a rate $\delta\frac{g_1-g_0}{g_0}$. However, this set is shrinking at a rate $g_1 > \delta\frac{g_1-g_0}{g_0}$ (because $z^*(t)$ grows at this rate). Therefore, the distribution of firms is given by

$$\begin{aligned} \mu(t,z) &= M(t - \frac{1}{g_1}\log(z))z^{-\frac{\delta}{g_1}} \qquad \text{if } z \leq z^*(t) \\ &= M_0 e^{t\delta\frac{g_1-g_0}{g_0}} z^{-\frac{\delta}{g_0}} \qquad \text{if } z > z^*(t) \end{aligned} \tag{32}$$

We now add back the exporters. Let g_{ij} be the rate of growth of a firm type $j = d, x$ (non exporter, exporter), at time $i = 0, 1$ (before and after the shock, respectively). Define t_1 as the time it takes the first firm born after the shock to become an exporter, that is, $e^{g_{1d}t_1} = z_{1x}$. For $t < t_1$, define $z^*(t) = e^{g_{1d}t}$ as the maximum productivity of a firm born after the shock. We know that $g_{1,x} > g_{0,x}$, $g_{1,d} < g_{0,d}$ and $z_{1,x} < z_{0,x}$.

Focus on non-exporters first.If $t \geq t_1$ it is clear that all non-exporters were born after the shock. The measure of new non-exporters is

$$\mu(t,z) = M(t - \frac{1}{g_{1,d}}\log(z))z^{-\frac{\delta}{g_{1,d}}} \qquad \text{if } t \geq t_1$$

If $t < t_1$ then some non-exporters were born before the shock, that is, $z^*(t) < z_{1,x}$. In this case the measure of non-exporters would be as (32). Therefore we have

$$\begin{aligned}\mu(t,z) &= M(t - \frac{1}{g_{1,d}}\log(z))z^{-\frac{\delta}{g_{1,d}}} \qquad \text{if } z < z_{1,x} \text{ and } t < t_1 \\ &= M(t - \frac{1}{g_{1,d}}\log(z))z^{-\frac{\delta}{g_{1,d}}} \qquad \text{if } z < z^*(t) < z_{1x} \text{ and } t < t_1 \\ &= M_0 e^{t\delta\frac{g_{1,d}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} \qquad \text{if } z^*(t) < z < z_{1x} \text{ and } t < t_1\end{aligned} \tag{33}$$

For the exporters we need to consider two areas. The first one is $[z_{1,x}, z_{0,x}]$, the set of firms that became exporters immediately after the shock, and the second is $[z_{0,x}, \infty]$. First we focus on the area $[z_{1,x}, z_{0,x}]$. This case has a small trick, when there is an unexpected shock in tariff at the initial instant the distribution of firms changes in a discontinuous way. The measure of firms by productivity does not change but the firms that export or not do. In particular, all firms with $z \in [z_{1,x}, z_{0,x}]$ that before the shock were producing only for the domestic market become exporters. Hence their productivity growth rate jumps from $g_{0,d}$ to $g_{1,x}$. This effect should be taken into account during the simulations. Define $t_2 = \frac{1}{g_{1,x}}[\log(z_{0,x}) - t_1 g_{1,d}]$, which is the time that it takes to a new firm to reach the upper boundary of this area. Notice that $t_2 - t_1$ is the time that it takes to a firm to transit the particular area. If $t > t_2$ clearly all firms in this area would be firms born after the shock. The age of this firm would be given $s = \frac{1}{g_{1,x}}[\log(z) - t_1 g_{1,d}] = \frac{1}{g_{1,x}}[\log(z) - \log(z_{1,x})] = \frac{1}{g_{1,x}}[\log(\frac{z}{z_{1,x}})]$. This would solve a PDE as in equation (30) but now to pin down F we need the initial condition $\mu(t, z_{1,x}) = M(t - \frac{1}{g_{1,x}}\log(z_{1,x}))z_{1,x}^{-\frac{\delta}{g_{1,d}}}$ or in a more convenient form $\frac{1}{\delta}\log[\mu(t, z_{1,x})] = \frac{1}{\delta}\log[M(t - \frac{1}{g_{1,d}}\log(z_{1,x}))] - \frac{1}{g_{1,d}}\log[z_{1,x}]$. Thus, using (30), F must satisfy

$$t + \frac{1}{\delta}\log[M(t - \frac{1}{g_{1,d}}\log(z_{1,x}))] - \frac{1}{g_{1,d}}\log[z_{1,x}] = F(t - \frac{\log(z_{1,x})}{g_{1,x}})$$

Which generates

$$F(x) = \frac{1}{\delta}\log\left[M\left(x + [\frac{1}{g_{1,x}} - \frac{1}{g_{1,d}}]\log(z_{1,x})\right)\right] + x - \left[\frac{1}{g_{1,d}} - \frac{1}{g_{1,x}}\right]\log(z_{1,x})$$

Using this in the general solution for the PDE we get

$$t + \frac{\log(\mu(t,z))}{\delta} = \frac{1}{\delta}\log\left[M\left(t - \frac{\log(z)}{g_{1,x}} + [\frac{1}{g_{1,x}} - \frac{1}{g_{1,d}}]\log(z_{1,x})\right)\right] + t - \frac{\log(z)}{g_{1,x}} - \left[\frac{1}{g_{1,d}} - \frac{1}{g_{1,x}}\right]\log(z_{1,x})$$

$$\mu(t,z) = M\left(t - \frac{1}{g_{1,x}}\log(z) + [\frac{1}{g_{1,x}} - \frac{1}{g_{1,d}}]\log(z_{1,x})\right) z_{1,x}^{-\frac{\delta}{g_{1,d}}}\left[\frac{z}{z_{1,x}}\right]^{-\frac{\delta}{g_{1,x}}} \quad (34)$$

The above equation is the measure of firms $z \in [z_{1,x}, z_{0,x}]$ when $t > t_2$. Notice that this would remain true for new firms with $z > z_{0,x}$. That is for every t the measure of exporters with productivity $z < z_{1,x}^{g_{1,x}(t-t_1)}$ will be given by equation (34).

If $t < t_1$, then no new firm has made it until this area, but there are two kinds of old firms. Those that were in the area before the shock and those who were non-exporters before and immediately after the shock but with time they have become exporters. Given this fact the number of old exporters in this area would be given for those old firms that are coming from the non-exporting sector and those firms that became exporters right after the shock and they haven't grown beyond $z_{0,x}$. (if the area is small it might happen that all initial exporters left already, check this later). Let $z^{1*}(t) = z_{1,x}e^{tg_{1,x}}$. If $z^{1*}(t) \leq z_{0,x}$ there are still firms in this area that were born before the shock and became immediately exporters. The measure of these firms would be given by (31), that is

$$\mu(t,z) = M_0 e^{t\delta\frac{g_{1,x}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} \quad \text{if } z^{1*}(t) < z \leq z_{0,x} \text{ and } t < t_1 \quad (35)$$

For the firms with $z \leq z^{1*}$ (the same applies if $z^{1*} > z_{0,x}$) they measure would solve the already seen PDE, that is

$$e^t(\mu)^{\frac{1}{\delta}} = F[e^t z^{-\frac{1}{g_{1,x}}}]$$

But now to pin down the the function F, the initial condition is $\mu(t, z_{1,x}) = M_0 e^{t\delta\frac{g_{1,d}-g_{0,d}}{g_{0,d}}} z_{1,x}^{-\frac{\delta}{g_{0,d}}}$, that is all old non-exporters that are becoming exporters. Which implies

$$e^t M_0^{\frac{1}{\delta}} e^{t\frac{g_{1,d}-g_{0,d}}{g_{0,d}}} z_{1,x}^{-\frac{1}{g_{0,d}}} = F[e^t z_{1,x}^{-\frac{1}{g_{1,x}}}]$$

$$M_0^{\frac{1}{\delta}} e^{t\frac{g_{1,d}}{g_{0,d}}} z_{1,x}^{-\frac{1}{g_{0,d}}} = F[e^t z_{1,x}^{-\frac{1}{g_{1,x}}}]$$

Then,

$$F(x) = M_0^{\frac{1}{\delta}} z_{1,x}^{\frac{1}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} x^{\frac{g_{1,d}}{g_{0,d}}}$$

Therefore,

$$e^t(\mu(t,z))^{\frac{1}{\delta}} = M_0^{\frac{1}{\delta}} z_{1,x}^{\frac{1}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} \left(e^t z^{-\frac{1}{g_{1,x}}}\right)^{\frac{g_{1,d}}{g_{0,d}}}$$

$$(\mu(t,z))^{\frac{1}{\delta}} = M_0^{\frac{1}{\delta}} z_{1,x}^{\frac{1}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} e^{t\left(\frac{g_{1,d}}{g_{0,d}}-1\right)} z^{-\frac{g_{1,d}}{g_{1,x}g_{0,d}}}$$

$$\mu(t,z) = M_0 z_{1,x}^{\frac{\delta}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} e^{t\delta\left(\frac{g_{1,d}}{g_{0,d}}-1\right)} z^{-\delta\frac{g_{1,d}}{g_{1,x}g_{0,d}}} \tag{36}$$

If $t \in [t_1, t_2]$, there are three possible types of firms in the area, new, old that were in the are before the shock and old that were non-exporters before shock and with time got into the area. In particular, a new firm cannot have a z larger than $z^{*2}(t) = e^{t_1 g_{1,d}+(t-t_1)g_{1,x}}$. Thus, all firms with $z \leq z^{*2}(t)$ will be new and will have a measure given by (34), while all firms with $z^{*2}(t) < z < z^{*1}(t)$ will be old firms that got there with time, will have a measure given by (36) and finally, firms with $z^{*1}(t) < z < z_{1,x}$ they were there before the shock and they still remain there (if this last inequality does no hold this measure would be zero). In short,

$$\begin{aligned}
\mu(t,z) &= M\left(t - \frac{1}{g_{1,x}}\log(z) + [\frac{1}{g_{1,x}} - \frac{1}{g_{1,d}}]\log(z_{1,x})\right) z_{1,x}^{-\frac{\delta}{g_{1,d}}} \left[\frac{z}{z_{1,x}}\right]^{-\frac{\delta}{g_{1,x}}} && \text{if } z_{1,x} \leq z \leq z_{0,x} \text{ and } t \geq t_2 \\
&= M\left(t - \frac{1}{g_{1,x}}\log(z) + [\frac{1}{g_{1,x}} - \frac{1}{g_{1,d}}]\log(z_{1,x})\right) z_{1,x}^{-\frac{\delta}{g_{1,d}}} \left[\frac{z}{z_{1,x}}\right]^{-\frac{\delta}{g_{1,x}}} && \text{if } z_{1,x} \leq z \leq z^{2*}(t) \text{ and } t_1 < t < t \\
&= M_0 z_{1,x}^{\frac{\delta}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} e^{t\delta\left(\frac{g_{1,d}}{g_{0,d}}-1\right)} z^{-\delta\frac{g_{1,d}}{g_{1,x}g_{0,d}}} && \text{if } z^{2*}(t) < z \leq z^{1*}(t) \text{ and } t_1 < t < t_2 \\
&= M_0 e^{t\delta\frac{g_{1,x}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^{1*}(t) < z \leq z_{0,x} \text{ and } t_1 < t < t_2 \\
&= M_0 z_{1,x}^{\frac{\delta}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} e^{t\delta\left(\frac{g_{1,d}}{g_{0,d}}-1\right)} z^{-\delta\frac{g_{1,d}}{g_{1,x}g_{0,d}}} && \text{if } z_{1,x} < z < z^{1*}(t) \text{ and } t < t_1 \\
&= M_0 e^{t\delta\frac{g_{1,x}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^{1*}(t) < z \leq z_{0,x} \text{ and } t < t_1
\end{aligned} \tag{37}$$

What happens with the measure of firms above $z_{0,x}$? First consider the firms that were already in this area before the shock. Clearly all firms with productivity $z_{0,x} \leq z < z_{0,x}e^{tg_{1,x}}$ already left or died. Thus, of the firms that were there before the shock only those with $z \geq z_{0,x}e^{tg_{1,x}} = z^{3*}(t)$ remain there, and their measure is similar to (31). The difference is that now the initial condition is

$$\mu(0,z) = M_0 z_{0,x}^{\frac{\delta}{g_{0,x}} - \frac{\delta}{g_{0,d}}} z^{-\frac{\delta}{g_{0x}}}$$

The resulting distribution is

$$\mu(t,z) = M_0 e^{t\delta\frac{g_{1,x}-g_{0,x}}{g_{0,x}}} z_{0,x}^{\frac{\delta}{g_{0,x}} - \frac{\delta}{g_{0,d}}} z^{-\frac{\delta}{g_{0,x}}} \quad \text{if } z \geq z^{3*}(t)$$

What about firms with $z_{0,x} \leq z < z^{3*}(t)$? This depends on t, if $t < t_2$ no new firm has made to this area. But if $t \geq t_2$ some new firms are in this area. In particular, they could have made it as high as $z^{2*}(t)$ only, and their measure would be given by (34). Hence, if $t \geq t_2$ we would have

$$\mu(t,z) = M\left(t - \frac{1}{g_{1,x}}\log(z) + [\frac{1}{g_{1,x}} - \frac{1}{g_{1,d}}]\log(z_{1,x})\right) z_{1,x}^{-\frac{\delta}{g_{1,d}}} \left[\frac{z}{z_{1,x}}\right]^{-\frac{\delta}{g_{1,x}}}$$

For all $z_{0,x} \leq z < z^{2*}(t)$.

In between there would be only old firms. Those that became exporters immediately after the shock and they made it beyond $z_{0,x}$ and those that were non-exporters at the very beginning and again they made it as well beyond $z_{0,x}$. Of course, if these two kinds of firms are present depends on t. If t is small only the first will be there and when t becomes larger the second start to show up. The first firms to start to show up would be those that were closer to $z_{0,x}$ before the shock. Hence, firms with $z^{1*}(t) \leq z < z^{3*}(t)$ will have a measure

$$\mu(t,z) = M_0 e^{t\delta\frac{g_{1,x}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} \quad \text{if } z^{1*}(t) \leq z < z^{3*}(t)$$

Then, old firms that remained non-exporters after the shock will appear, with measure

$$M_0 z_{1,x}^{\frac{\delta}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} e^{t\delta\left(\frac{g_{1,d}}{g_{0,d}}-1\right)} z^{-\delta\frac{g_{1,d}}{g_{1,x}g_{0,d}}} \quad \text{if } z^{2*}(t) \leq z < z^{1*}(t)$$

Therefore, the measures in this area will be given by

$$\begin{aligned}
\mu(t,z) &= M_0 e^{t\delta\frac{g_{1,x}-g_{0,x}}{g_{0,x}}} z^{-\frac{\delta}{g_{0,x}}} && \text{if } z \geq z^{3*}(t) \forall t \\
&= M_0 e^{t\delta\frac{g_{1,x}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^{1*}(t) < z \leq z^{3*}(t) \forall t \\
&= M_0 z_{1,x}^{\frac{\delta}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} e^{t\delta\left(\frac{g_{1,d}}{g_{0,d}}-1\right)} z^{-\delta\frac{g_{1,d}}{g_{1,x}g_{0,d}}} && \text{if } z^{2*}(t) < z < z^{1*}(t) t > t_1 \\
&= M\left(t - \frac{1}{g_{1,x}}\log(z) + [\frac{1}{g_{1,x}} - \frac{1}{g_{1,d}}]\log(z_{1,x})\right) z_{1,x}^{-\frac{\delta}{g_{1,d}}} \left[\frac{z}{z_{1,x}}\right]^{-\frac{\delta}{g_{1,x}}} && \text{if } z_{0,x} \leq z \leq z^{2*}(t) \text{ and } t \geq t_2
\end{aligned} \tag{38}$$

Re-writing the firms distribution

If $t = 0$ the distribution of firms is the same as before the shock with the only difference that the number of exporters is given by $z > z_{1,x}$ instead of $z > z_{0,x}$. This initial jump allows us to pin down $M(0)$ (not M_0) and $I(0)$ given the new level of Π_d.

Define $t_1 : e^{g_d t_1} = z_{x1}, z^*(t) = e^{g_{1,d}t}$, $z^{1*}(t) = \min\{e^{g_{1,x}t} z_{1,x}, z_{0x}\}$ (if $z^{1*}(t) > z_{0,x}$ set $z^{1*}(t) = z_{0,x}$), $z^{3*}(t) = e^{g_{1,x}t} z_{0,x}, z^{*2}(t) = e^{t_1 g_{1,d} + (t-t_1)g_{1,x}}$

If $t < t_1$,

$$\begin{aligned}
\mu(t,z) &= M(t - \frac{1}{g_{1,d}}\log(z)) z^{-\frac{\delta}{g_{1,d}}} && \text{if } z \leq z^*(t) \\
&= M_0 e^{t\delta\frac{g_{1,d}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^*(t) < z \leq z_{1,x} \\
&= M_0 z_{1,x}^{\frac{\delta}{g_{0,d}}\left(\frac{g_{1,d}}{g_{1,x}}-1\right)} e^{t\delta\left(\frac{g_{1,d}}{g_{0,d}}-1\right)} z^{-\delta\frac{g_{1,d}}{g_{1,x}g_{0,d}}} && \text{if } z_{1,x} < z \leq z^{1*}(t) \\
&= M_0 e^{t\delta\frac{g_{1,x}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z^{1*}(t) < z \leq z_{0,x} \\
&= M_0 e^{t\delta\frac{g_{1,x}-g_{0,d}}{g_{0,d}}} z^{-\frac{\delta}{g_{0,d}}} && \text{if } z_{0,x} < z \leq z^{3*}(t) \\
&= M_0 e^{t\delta\frac{g_{1,x}-g_{0,x}}{g_{0,x}}} z^{-\frac{\delta}{g_{0,x}}} && \text{if } z > z^{3*}(t)
\end{aligned}$$

The way to identify M is using the labor market clearing condition. This is

$$\begin{aligned}
1 =& \Pi_d(t)\left[\int_1^{z_x(t)} z\mu(t,z)dz + (1 + (1+\tau)^{-2})\int_{z_x(t)}^{\infty} z\mu(t,z)dz\right] + \\
& \frac{\kappa_1}{2}\left[g_d(t)^2 \int_1^{z_x(t)} z\mu(t,z)dz + g_x(t)^2 \int_{z_x(t)}^{\infty} z\mu(t,z)dz + \kappa_e M(t)\right]
\end{aligned}$$

Or alternatively, assuming $z^{1*}(t) > z_{0x}$, for $t < t_1$,

$$\frac{1}{\Pi_d(t)} = (1+\kappa_1 g_d)\left[\int_1^{z^*(t)} z\mu(t,z)dz + \int_{z^*(t)}^{z_{x1}} z\mu(t,z)dz\right] +$$
$$\left(1+(1+\tau)^{-2}+\kappa_1 g_x\right)\times$$
$$\left[\int_{z_{1x}}^{z^{1*}(t)} z\mu(t,z)dz + \int_{z^{1*}(t)}^{z_{0x}} z\mu(t,z)dz + \int_{z_{0x}}^{z^{3*}(t)} z\mu(t,z)dz + \int_{z^{3*}(t)}^{\infty} z\mu(t,z)dz + \kappa_e M(t)\right]$$

Integral 2

$$\int_{z^*(t)}^{z_{x1}} M_0 e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z^{1-\frac{\delta}{g_{0d}}} dz = \frac{M_0}{2-\frac{\delta}{g_{0d}}}\left(e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z_{x1}^{2-\frac{\delta}{g_{0d}}} - e^{t(2g_{1d}-\delta)}\right)$$

Integral 3

$$\int_{z_{x1}}^{z^1(t)} M_0 z_{x1}^{\frac{\delta}{g_{0d}}\left(\frac{g_{1d}-g_{1x}}{g_{1x}}\right)} e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z^{1-\frac{\delta g_{1d}}{g_{1x}g_{0d}}} dz = \frac{M_0 z_{1x}^{2-\frac{\delta}{g_{0d}}} e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)}}{2-\frac{\delta g_{1d}}{g_{1x}g_{0d}}}\left[e^{t(2g_{1x}-\frac{\delta g_{1d}}{g_{0d}})} - 1\right]$$
$$= \frac{M_0 z_{1x}^{2-\frac{\delta}{g_{0d}}}}{2-\frac{\delta g_{1d}}{g_{1x}g_{0d}}}\left[e^{t(2g_{1x}-\delta)} - e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)}\right]$$

Integral 4

$$\int_{z^1(t)}^{z^3(t)} M_0 e^{t\delta\left(\frac{g_{1x}-g_{0d}}{g_{0d}}\right)} z^{1-\frac{\delta}{g_{0d}}} dz = \frac{M_0 e^{t(2g_{1x}-\delta)}}{2-\frac{\delta}{g_{0d}}}\left[z_{0x}^{2-\frac{\delta}{g_{0d}}} - z_{1x}^{2-\frac{\delta}{g_{0d}}}\right]$$

Integral 5

$$\int_{z^3(t)}^{\infty} M_0 e^{t\delta\left(\frac{g_{1x}-g_{0x}}{g_{0x}}\right)} z^{1-\frac{\delta}{g_{0x}}} dz = \frac{M_0 z_{0x}^{2-\frac{\delta}{g_{0d}}}}{-2+\frac{\delta}{g_{0x}}} e^{t(2g_{1x}-\delta)}$$

Notice that the average productivity of exporters is given by

$$E^x(z) = \frac{M_0 z_{1x}^{2-\frac{\delta}{g_{0d}}}}{2-\frac{\delta g_{1d}}{g_{1x}g_{0d}}}\left[e^{t(2g_{1x}-\delta)} - e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)}\right] + \frac{M_0 e^{t(2g_{1x}-\delta)}}{2-\frac{\delta}{g_{0d}}}\left[z_{0x}^{2-\frac{\delta}{g_{0d}}} - z_{1x}^{2-\frac{\delta}{g_{0d}}}\right] + \frac{M_0 z_{0x}^{2-\frac{\delta}{g_{0x}}}}{-2+\frac{\delta}{g_{0x}}} e^{t(2g_{1x}-\delta)}$$

then

$$E^x(t,z) = \psi_0 e^{\psi_1 t} + \psi_2 e^{\psi_3 t} \tag{39}$$

where

$$\psi_0 = M_0 \left[\frac{\left(z_{0x}^{2-\frac{\delta}{g_{0d}}} - z_{1x}^{2-\frac{\delta}{g_{0d}}} \right)}{2 - \frac{\delta}{g_{0d}}} - \frac{z_{0x}^{2-\frac{\delta}{g_{0x}}}}{2 - \frac{\delta}{g_{0x}}} + \frac{z_{1x}^{2-\frac{\delta}{g_{0d}}}}{2 - \frac{\delta g_{1d}}{g_{1x} g_{0d}}} \right]$$

$$\psi_1 = 2g_{1x} - \delta < 0$$

$$\psi_2 = -\frac{M_0 z_{1x}^{2-\frac{\delta}{g_{0d}}}}{2 - \frac{\delta g_{1d}}{g_{1x} g_{0d}}}$$

$$\psi_3 = \delta \left(\frac{g_{1d} - g_{0d}}{g_{0d}} \right) < 0$$

The average productivity of non-exporters is given by

$$E^n(t,z) = \int_1^{z^*(t)} \mu(t,z) z dz + \frac{M_0}{2 - \frac{\delta}{g_{0d}}} \left(e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z_{x1}^{2-\frac{\delta}{g_{0d}}} - e^{t(2g_{1d}-\delta)} \right)$$

$$E^n(t,z) = \int_1^{z^*(t)} \mu(t,z) z dz + E^{n1}(t,z)$$

$$E^n(t,z) = E^{n0}(t,z) + E^{n1}(t,z)$$

$$E^n(z) = E^{n0}(z) + \psi_4 e^{\psi_3 t} + \psi_5 e^{\psi_6 t}$$

where

$$\psi_4 = \frac{M_0 z_{x1}^{2-\frac{\delta}{g_{0d}}}}{2 - \frac{\delta}{g_{0d}}}$$

$$\psi_5 = -\frac{M_0}{2 - \frac{\delta}{g_{0d}}}$$

$$\psi_6 = 2g_{1d} - \delta$$

Then we can write market clearing as

$$1 = \Pi_{d1}\left[\int_1^{z_{1x}} z\mu(t,z)dz + (1+(1+\tau)^{-2})\int_{z_{1x}}^{\infty} z\mu(t,z)dz\right] + \frac{\kappa_1}{2}\left[g_{1d}^2\int_1^{z_{1x}} z\mu(t,z)dz + g_{1x}^2\int_{z_{1x}}^{\infty} z\mu(t,z)dz\right] + \kappa_e M(t)$$

$$1 = \left[\Pi_{d1} + \frac{\kappa_1 g_{1d}^2}{2}\right]\int_1^{z_{1x}} z\mu(t,z)dz + \left[\Pi_{d1}(1+(1+\tau)^{-2}) + \frac{\kappa_1 g_{1x}^2}{2}\right]\int_{z_{1x}}^{\infty} z\mu(t,z)dz + \kappa_e M(t)$$

$$1 = \left[\Pi_{d1} + \frac{\kappa_1 g_{1d}^2}{2}\right][E^{n0}(t,z) + E^{n1}(t,z)] + \left[\Pi_{d1}(1+(1+\tau)^{-2}) + \frac{\kappa_1 g_{1x}^2}{2}\right]E^{x}(t,z) + \kappa_e M(t)$$

Let $a_0 = \Pi_{d1} + \frac{\kappa_1 g_{1d}^2}{2}$ and $a_1 = \Pi_{d1}(1+(1+\tau)^{-2}) + \frac{\kappa_1 g_{1x}^2}{2}$, then

$$1 = a_0[E^{n0}(t,z) + \psi_4 e^{\psi_3 t} + \psi_5 e^{\psi_6 t}] + a_1[\psi_0 e^{\psi_1 t} + \psi_2 e^{\psi_3 t}] + \kappa_e M(t)$$

Then we must have

$$E^{n0}(t,z) = \frac{1}{a_0} - \psi_4 e^{\psi_3 t} - \psi_5 e^{\psi_6 t} - \frac{a_1}{a_0}[\psi_0 e^{\psi_1 t} + \psi_2 e^{\psi_3 t}] - \frac{\kappa_e}{a_0}M(t) \tag{40}$$

For this reason we proposed the following function for $M(t)$

$$M(t) = m_0 + m_1 e^{m_2 t} + m_3 e^{m_4 t}$$

Integrating the remaining part we get

$$
\begin{aligned}
&E^{n0}(t,z) = \int_1^{z^*(t)} z\mu(t,z)dz \\
&= -\frac{m_0}{2-\frac{\delta}{g_{1d}}} + \left[\frac{m_0}{2-\frac{\delta}{g_{1d}}} + \frac{m_1}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3}{2-\frac{\delta+m_4}{g_{1d}}}\right] e^{t\psi_6} - \left[\frac{m_1 e^{tm_2}}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3 e^{tm_4}}{2-\frac{\delta+m_4}{g_{1d}}}\right]
\end{aligned}
$$

We have five unknowns, m's, but we have five ψ's that we can use to pin them dow. First, we can arbitrarily set $m_4 = \psi_3$ and $m_2 = \psi_1$. Then we can write the above equation as

$$
E^{n0}(t,z) = -\frac{m_0}{2-\frac{\delta}{g_{1d}}} + \left[\frac{m_0}{2-\frac{\delta}{g_{1d}}} + \frac{m_1}{2-\frac{\delta+\psi_1}{g_{1d}}} + \frac{m_3}{2-\frac{\delta+\psi_3}{g_{1d}}}\right] e^{t\psi_6} - \left[\frac{m_1 e^{t\psi_1}}{2-\frac{\delta+\psi_1}{g_{1d}}} + \frac{m_3 e^{t\psi_3}}{2-\frac{\delta+\psi_3}{g_{1d}}}\right]
$$

and rewriting (40) we have

$$
E^{n0}(t,z) = \frac{1}{a_0} - \frac{\kappa_e}{a_0}m_0 - \left[\psi_4 + \frac{a_1}{a_0}\psi_2 + \frac{\kappa_e}{a_0}m_3\right] e^{\psi_3 t} - \psi_5 e^{\psi_6 t} - \left[\frac{a_1}{a_0}\psi_0 + \frac{\kappa_e}{a_0}m_1\right] e^{\psi_1 t}
$$

From here we immediately get

$$
\begin{aligned}
-\frac{m_0}{2-\frac{\delta}{g_{1d}}} &= \frac{1}{a_0} - \frac{\kappa_e}{a_0}m_0 \\
\frac{m_1}{2-\frac{\delta+\psi_1}{g_{1d}}} &= \frac{a_1}{a_0}\psi_0 + \frac{\kappa_e}{a_0}m_1 \\
\frac{m_3}{2-\frac{\delta+\psi_3}{g_{1d}}} &= \left[\psi_4 + \frac{a_1}{a_0}\psi_2 + \frac{\kappa_e}{a_0}m_3\right]
\end{aligned}
$$

Now, we need ψ_5 but we have no m left. However, notice that the above assignment would be a solution only if,

$$
\psi_5 = \frac{1}{a_0} - \frac{\kappa_e}{a_0}m_0 - \frac{a_1}{a_0}\psi_0 - \frac{\kappa_e}{a_0}m_1 - \left[\psi_4 + \frac{a_1}{a_0}\psi_2 + \frac{\kappa_e}{a_0}m_3\right]
$$

But, the above equality is implied by equation (40) when $t = 0$.

For $t_1 < t < 2t_1$

$$
\begin{aligned}
\mu(t,z) &= M\left(t - \frac{\log(z)}{g_{1d}}\right) z^{-\frac{\delta}{g_{1d}}} \qquad \text{if } z \leq z_{1x} \\
&= M\left(t - \frac{\log(z)}{g_{1x}} + \left(\frac{1}{g_{1x}} - \frac{1}{g_{1d}}\right)\log(z_{1x})\right) z_{1x}^{\delta\left(\frac{1}{g_{1x}} - \frac{1}{g_{1d}}\right)} z^{-\frac{\delta}{g_{1x}}} \qquad \text{if } z_{1x} \leq z \leq z^{2*}(t) \\
&= M_0 z_{1x}^{\frac{\delta}{g_{0d}}\left(\frac{g_{1d}-g_{1x}}{g_{1x}}\right)} e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z^{-\delta\frac{g_{1d}}{g_{1x}g_{0d}}} \qquad \text{if } z^{2*}(t) \leq z \leq z^{1*}(t) \\
&= M_0 e^{t\delta\left(\frac{g_{1x}-g_{0d}}{g_{0d}}\right)} z^{-\frac{\delta}{g_{0d}}} \qquad \text{if } z^{1*}(t) \leq z \leq z^{3*}(t) \\
&= M_0 e^{t\delta\left(\frac{g_{1x}-g_{0x}}{g_{0x}}\right)} z^{-\frac{\delta}{g_{0x}}} \qquad \text{if } z > z^{3*}(t)
\end{aligned}
$$

Integral 3

$$
\int_{z^{2*}(t)}^{z^{1*}(t)} M_0 z_{1x}^{\frac{\delta}{g_{0d}}\left(\frac{g_{1d}-g_{1x}}{g_{1x}}\right)} e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z^{1-\delta\frac{g_{1d}}{g_{1x}g_{0d}}}\, dz = e^{t(2g_{1x}-\delta)} \frac{M_0 z_{1x}^{2-\frac{\delta}{g_{0d}}}}{2 - \frac{\delta/g_{1d}}{g_{1x}g_{0d}}} \left(1 - e^{-t_1\left(2g_{1x} - \delta\frac{g_{1d}}{g_{0d}}\right)}\right)
$$

Integral 4

$$
\int_{z^{1*}(t)}^{z^{3*}(t)} M_0 e^{t\delta\left(\frac{g_{1x}-g_{0d}}{g_{0d}}\right)} z^{1-\frac{\delta}{g_{0d}}}\, dz = \frac{M_0 e^{t(2g_{1x}-\delta)}}{2-\delta/g_{0d}} \left(z_{0x}^{2-\delta/g_{0d}} - z_{1x}^{2-\delta/g_{0d}}\right)
$$

Integral 5

$$
\int_{z^{3*}(t)}^{\infty} M_0 e^{t\delta\left(\frac{g_{1x}-g_{0x}}{g_{0x}}\right)} z^{1-\frac{\delta}{g_{0x}}}\, dz = \frac{M_0 e^{t(2g_{1x}-\delta)}}{-2+\delta/g_{0x}} z_{0x}^{2-\delta/g_{0d}}
$$

Define

$$\tilde{E}_x(t) = \int_{z^{2*}(t)}^{z^{1*}(t)} M_0 z_{1x}^{\frac{\delta}{g_{0d}}\left(\frac{g_{1d}-g_{1x}}{g_{1x}}\right)} e^{t\delta\left(\frac{g_{1d}-g_{0d}}{g_{0d}}\right)} z^{1-\delta\frac{g_{1d}}{g_{1x}g_{0d}}} dz +$$
$$\int_{z^{1*}(t)}^{z^{3*}(t)} M_0 e^{t\delta\left(\frac{g_{1x}-g_{0d}}{g_{0d}}\right)} z^{1-\frac{\delta}{g_{0d}}} dz +$$
$$\int_{z^{3*}(t)}^{\infty} M_0 e^{t\delta\left(\frac{g_{1x}-g_{0x}}{g_{0x}}\right)} z^{1-\frac{\delta}{g_{0x}}} dz$$
$$\tilde{E}_x(t) = e^{t(2g_{1x}-\delta)} \frac{M_0 z_{1x}^{2-\frac{\delta}{g_{0d}}}}{2-\frac{\delta g_{1d}}{g_{1x}g_{0d}}} \left(1-e^{-t_1\left(2g_{1x}-\delta\frac{g_{1d}}{g_{0d}}\right)}\right)$$
$$\frac{M_0 e^{t(2g_{1x}-\delta)}}{2-\delta/g_{0d}} \left(z_{0x}^{2-\delta/g_{0d}} - z_{1x}^{2-\delta/g_{0d}}\right)$$
$$\frac{M_0 e^{t(2g_{1x}-\delta)}}{-2+\delta/g_{0x}} z_{0x}^{2-\delta/g_{0x}}$$

Then

$$\tilde{E}_x(t) = \psi_7 e^{t\psi_1}$$

where

$$\psi_7 = M_0 \left[\frac{z_{1x}^{2-\frac{\delta}{g_{0d}}}}{2-\frac{\delta g_{1d}}{g_{1x}g_{0d}}} \left(1-e^{-t_1\left(2g_{1x}-\delta\frac{g_{1d}}{g_{0d}}\right)}\right) + \frac{z_{0x}^{2-\delta/g_{0d}} - z_{1x}^{2-\delta/g_{0d}}}{2-\delta/g_{0d}} + \frac{z_{0x}^{2-\delta/g_{0x}}}{-2+\delta/g_{0x}} \right]$$

Notice

$$\psi_7 = \psi_0 + \psi_2 e^{-t_1(\psi_1-\psi_3)}$$

Next we need to solve the integrals where $M(t)$ appears. One thing to notice is that we know $M(t)$ for some new firms, the ones born before t_1. These can be exporters or non exporters. Since it takes t_1 periods to become an exporter, when $t < 2t_1$, some new firms born before t_1 are non exporters. For $t > 2t_1$, all new firms born before t_1 are exporting.

Define $z_4(t)$ as the productivity of a firm born in t_1.

$$z_4(t) = \begin{cases} e^{g_{1d}(t-t_1)} \text{ if } t < 2t_1 \\ z_{x1} e^{g_{x1}(t-2t_1)} \text{ if } t > 2t_1 \end{cases}$$

For $t < 2t_1$

$$\int_1^{z_{1x}} z\mu(t,z)dz = \int_1^{z_4(t)} z\mu(t,z)dz + \int_{z_4(t)}^{z_1} z\mu(t,z)dz$$

Let

$$\tilde{E}_d(t) = \int_{z_4(t)}^{z_{x1}} z\mu(t,z)dz$$

$$\begin{aligned}
\int_{z_4(t)}^{z_{1x}} z\mu(t,z)dz =& \int_{z_4(t)}^{z_1} M\left(t - \frac{\log z}{g_{1d}}\right) z^{1-\frac{\delta}{g_{1d}}} dz = \int_{z_4(t)}^{z_1} \left[m_0 z^{1-\frac{\delta}{g_{1d}}} + m_1 e^{m_2 t} z^{1-\frac{\delta+m_2}{g_{1d}}} + \dots\right] dz \\
=& \frac{m_0}{2-\frac{\delta}{g_{1d}}}\left(z_{1x}^{2-\frac{\delta}{g_{1d}}} - z_4(t)^{2-\frac{\delta}{g_{1d}}}\right) + \frac{m_1 e^{m_2 t}}{2-\frac{\delta+m_2}{g_{1d}}}\left(z_{1x}^{2-\frac{\delta+m_2}{g_{1d}}} - z_4(t)^{2-\frac{\delta+m_2}{g_{1d}}}\right) + \dots \\
=& \frac{m_0}{2-\frac{\delta}{g_{1d}}}\left(z_{1x}^{2-\frac{\delta}{g_{1d}}} - e^{g_{1d}(t-t_1)\left(2-\frac{\delta}{g_{1d}}\right)}\right) + \frac{m_1 e^{m_2 t}}{2-\frac{\delta+m_2}{g_{1d}}}\left(z_{1x}^{2-\frac{\delta+m_2}{g_{1d}}} - e^{g_{1d}(t-t_1)\left(2-\frac{\delta+m_2}{g_{1d}}\right)}\right) + \dots \\
=& \frac{m_0}{2-\frac{\delta}{g_{1d}}}\left(z_{1x}^{2-\frac{\delta}{g_{1d}}} - e^{(t-t_1)(2g_{1d}-\delta)}\right) + \frac{m_1 e^{m_2 t}}{2-\frac{\delta+m_2}{g_{1d}}}\left(z_{1x}^{2-\frac{\delta+m_2}{g_{1d}}} - e^{(t-t_1)(2g_{1d}-\delta-m_2)}\right) + \dots \\
=& \frac{m_0}{2-\frac{\delta}{g_{1d}}}\left(z_{1x}^{2-\frac{\delta}{g_{1d}}} - e^{(t-t_1)(2g_{1d}-\delta)}\right) + \frac{m_1}{2-\frac{\delta+m_2}{g_{1d}}}\left(e^{m_2 t} z_{1x}^{2-\frac{\delta+m_2}{g_{1d}}} - e^{t(2g_{1d}-\delta)} e^{-t_1(2g_{1d}-\delta-m_2)}\right) +
\end{aligned}$$

$$\begin{aligned}
\int_{z_4(t)}^{z_{1x}} z\mu(t,z)dz =& \frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1d}}}}{2-\frac{\delta}{g_{1d}}} - e^{t\psi_6}\left(\frac{m_0 e^{-t_1\psi_6}}{2-\frac{\delta}{g_{1d}}} + \frac{m_1 e^{-t_1(2g_{1d}-\delta-m_2)}}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3 e^{-t_1(2g_{1d}-\delta-m_4)}}{2-\frac{\delta+m_4}{g_{1d}}}\right) \\
&+ \frac{m_1 z_{1x}^{2-\frac{\delta+m_2}{g_{1d}}} e^{m_2 t}}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3 z_{1x}^{2-\frac{\delta+m_4}{g_{1d}}} e^{m_4 t}}{2-\frac{\delta+m_4}{g_{1d}}}
\end{aligned}$$

Since $\psi_6 = 2g_{1d} - \delta$

$$\begin{aligned}
\int_{z_4(t)}^{z_{1x}} z\mu(t,z)dz =& \frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1d}}}}{2-\frac{\delta}{g_{1d}}} - e^{t\psi_6}\left(\frac{m_0 e^{-t_1\psi_6}}{2-\frac{\delta}{g_{1d}}} + \frac{m_1 e^{-t_1(2g_{1d}-\delta-m_2)}}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3 e^{-t_1(2g_{1d}-\delta-m_4)}}{2-\frac{\delta+m_4}{g_{1d}}}\right) \\
&+ \frac{m_1 z_{1x}^{2-\frac{\delta+m_2}{g_{1d}}} e^{m_2 t}}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3 z_{1x}^{2-\frac{\delta+m_4}{g_{1d}}} e^{m_4 t}}{2-\frac{\delta+m_4}{g_{1d}}} \\
=& \frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1d}}}}{2-\frac{\delta}{g_{1d}}} - e^{t\psi_6}e^{-t_1\psi_6}\left(\frac{m_0}{2-\frac{\delta}{g_{1d}}} + \frac{m_1 e^{t_1 m_2}}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3 e^{t_1 m_4}}{2-\frac{\delta+m_4}{g_{1d}}}\right) \\
&+ \frac{m_1 z_{1x}^{2-\frac{\delta+m_2}{g_{1d}}} e^{m_2 t}}{2-\frac{\delta+m_2}{g_{1d}}} + \frac{m_3 z_{1x}^{2-\frac{\delta+m_4}{g_{1d}}} e^{m_4 t}}{2-\frac{\delta+m_4}{g_{1d}}}
\end{aligned}$$

Since $m_2 = \psi_1$ and $m_4 = \psi_3$

$$\begin{aligned}
\int_{z_4(t)}^{z_{1x}} z\mu(t,z)dz =& \frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1d}}}}{2-\frac{\delta}{g_{1d}}} - e^{t\psi_6}e^{-t_1\psi_6}\left(\frac{m_0}{2-\frac{\delta}{g_{1d}}} + \frac{m_1 e^{t_1 \psi_1}}{2-\frac{\delta+\psi_1}{g_{1d}}} + \frac{m_3 e^{t_1 \psi_3}}{2-\frac{\delta+\psi_3}{g_{1d}}}\right) \\
&+ \frac{m_1 z_{1x}^{2-\frac{\delta+\psi_1}{g_{1d}}}}{2-\frac{\delta+\psi_1}{g_{1d}}} e^{\psi_1 t} + \frac{m_3 z_{1x}^{2-\frac{\delta+\psi_3}{g_{1d}}}}{2-\frac{\delta+\psi_3}{g_{1d}}} e^{\psi_3 t}
\end{aligned}$$

then

$$E_{n1}(t) = \psi_{-1n} + \tilde{\psi}_8 e^{\psi_1 t} + \tilde{\psi}_4 e^{\psi_3 t} + \tilde{\psi}_5 e^{\psi_6 t}$$

where

$$\begin{aligned}
\tilde{\psi}_{-1n} =& \frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1d}}}}{2-\frac{\delta}{g_{1d}}} \\
\tilde{\psi}_4 =& \frac{m_3 z_{1x}^{2-\frac{\delta+\psi_3}{g_{1d}}}}{2-\frac{\delta+\psi_3}{g_{1d}}} \\
\tilde{\psi}_5 =& -e^{-t_1\psi_6}\left(\frac{m_0}{2-\frac{\delta}{g_{1d}}} + \frac{m_1 e^{t_1 \psi_1}}{2-\frac{\delta+\psi_1}{g_{1d}}} + \frac{m_3 e^{t_1 \psi_3}}{2-\frac{\delta+\psi_3}{g_{1d}}}\right) \\
\psi_8 =& \frac{m_1 z_{1x}^{2-\frac{\delta+\psi_1}{g_{1d}}}}{2-\frac{\delta+\psi_1}{g_{1d}}}
\end{aligned}$$

And,

$$E_x(t) = \int_{z_{1x}}^{z^{2*}(t)} z\mu(t,z)dz + \tilde{E}_x(t)$$

$$\int_{z_{1x}}^{z^{2*}(t)} z\mu(t,z)dz = \int_{z_{1x}}^{z_{1x}e^{(t-t_1)g_{1x}}} z_{1x}^{\delta\left(\frac{1}{g_{1x}}-\frac{1}{g_{1x}}\right)} z^{1-\frac{\delta}{g_{1x}}} M\left(t - \frac{\log z}{g_{1x}} + \left(\frac{1}{g_{1x}} - \frac{1}{g_{1d}}\right)\log z_{1x}\right) dz =$$
$$\frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1x}}}}{2-\frac{\delta}{g_{1x}}}\left[e^{(t-t_1)(2g_{1x}-\delta)} - 1\right] + \sum_{i=1,3} e^{tm_{i+1}} \frac{m_i z_{1x}^{(\delta+m_{i+1})\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}\right)} z_{1x}^{2g_{1x}-(\delta+m_{i+1})}}{2-\frac{\delta+m_{i+1}}{g_{1x}}}\left[e^{(t-t_1)(2g_{1x}-(\delta+m_{i+1}))} - 1\right]$$

notice that when $2g_{1x} - (\delta + m_2) = 0$, therefore

$$\int_{z_{1x}}^{z^{2*}(t)} z\mu(t,z)dz =$$
$$\frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1x}}}}{2-\frac{\delta}{g_{1x}}}\left[e^{(t-t_1)(2g_{1x}-\delta)} - 1\right] + \frac{e^{tm_4} m_3 z_{1x}^{(\delta+m_4)\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}\right)+2g_{1x}-(\delta+m_4)}}{2-\frac{\delta+m_4}{g_{1x}}}\left[e^{(t-t_1)(2g_{1x}-(\delta+m_4))} - 1\right]$$

$$\int_{z_{1x}}^{z^{2*}(t)} z\mu(t,z)dz = -\frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1x}}}}{2-\frac{\delta}{g_{1x}}} + \frac{e^{-t_1\psi_1} m_0 z_{1x}^{2-\frac{\delta}{g_{1x}}}}{2-\frac{\delta}{g_{1x}}} e^{t\psi_1} +$$
$$\frac{e^{-t_1(\psi_1-m_4)} m_3 z_{1x}^{(\delta+m_4)\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}\right)+2g_{1x}-(\delta+m_4)}}{2-\frac{\delta+m_4}{g_{1x}}} e^{t\psi_1} - \frac{m_3 z_{1x}^{(\delta+m_4)\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}\right)+2g_{1x}-(\delta+m_4)}}{2-\frac{\delta+m_4}{g_{1x}}} e^{tm_4}$$

Since $m_4 = \psi_3$

$$\int_{z_{1x}}^{z^{2*}(t)} z\mu(t,z)dz =$$
$$-\frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1x}}}}{2-\frac{\delta}{g_{1x}}} + e^{-t_1\psi_1}\left[\frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1x}}}}{2-\frac{\delta}{g_{1x}}} + \frac{e^{t_1\psi_3} m_3 z_{1x}^{(\delta+\psi_3)\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}-1\right)+2g_{1x}}}{2-\frac{\delta+\psi_3}{g_{1x}}}\right] e^{t\psi_1} - \frac{m_3 z_{1x}^{(\delta+\psi_3)\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}-1\right)+2g_{1x}}}{2-\frac{\delta+\psi_3}{g_{1x}}} e^{t\psi_3}$$

Thus the average productivity of exporters is

$$E^x(t,z) = \psi_{-1} + \tilde{\psi}_0 e^{\psi_1 t} + \tilde{\psi}_2 e^{\psi_3 t} \tag{41}$$

where

$$\psi_{-1} = -\frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1d}}}}{2-\frac{\delta}{g_{1x}}}$$

$$\tilde{\psi}_0 = \psi_0 + e^{-t_1\psi_1}\left[\frac{m_0 z_{1x}^{2-\frac{\delta}{g_{1d}}}}{2-\frac{\delta}{g_{1x}}} + \frac{e^{t_1\psi_3} m_3 z_{1x}^{(\delta+\psi_3)\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}-1\right)+2g_{1x}}}{2-\frac{\delta+\psi_3}{g_{1x}}} + \psi_2 e^{t_1\psi_3}\right]$$

$$\tilde{\psi}_2 = -\frac{m_3 z_{1x}^{(\delta+\psi_3)\left(\frac{1}{g_{1x}}-\frac{1}{g_{1d}}-1\right)+2g_{1x}}}{2-\frac{\delta+\psi_3}{g_{1x}}}$$

Thus,

$$\tilde{\psi}_0 = \psi_0 + e^{-t_1\psi_1}\left[-\psi_{-1} - e^{t_1\psi_3}\tilde{\psi}_2 - \psi_2 e^{t_1\psi_3}\right]$$

Therefore, market clearing is

$$1 = a_0[E^{n0}(t,z) + \psi_{-1n} + \tilde{\psi}_8 e^{\psi_1 t} + \tilde{\psi}_4 e^{\psi_3 t} + \tilde{\psi}_5 e^{\psi_6 t}] + a_1[\psi_{-1} + \tilde{\psi}_0 e^{\psi_1 t} + \tilde{\psi}_2 e^{\psi_3 t}] + \kappa_e M(t)$$

Then

$$E^{n0}(t,z) = \left(\frac{1}{a_0} - \psi_{-1n} - \frac{a_1}{a_0}\psi_{-1}\right) - \left(\tilde{\psi}_8 e^{\psi_1 t} + \tilde{\psi}_4 e^{\psi_3 t} + \tilde{\psi}_5 e^{\psi_6 t} + \frac{a_1}{a_0}[\tilde{\psi}_0 e^{\psi_1 t} + \tilde{\psi}_2 e^{\psi_3 t}]\right) - \frac{\kappa_e}{a_0} M(t)$$

$$E^{n0}(t,z) = \int_1^{z^4(t)} z\mu(t,z)dz = \int_1^{z^4(t)} z^{1-\frac{\delta}{g_{1d}}} M\left(t - \frac{\log z}{g_{1d}}\right) dz$$

Propose

$$M(t) = \tilde{m}_0 + \tilde{m}_1 e^{\tilde{m}_2 t} + \tilde{m}_3 e^{\tilde{m}_4 t}$$

$$E^{n0}(t,z) = \int_1^{z^4(t)} z^{1-\frac{\delta}{g_{1d}}} M\left(t - \frac{\log z}{g_{1d}}\right) dz =$$

$$\frac{\tilde{m}_0 \left(z^4(t)^{2-\frac{\delta}{g_{1d}}} - 1\right)}{2 - \frac{\delta}{g_{1d}}} + \frac{\tilde{m}_1 e^{t\tilde{m}_2} \left(z^4(t)^{2-\frac{\delta+\tilde{m}_2}{g_{1d}}} - 1\right)}{2 - \frac{\delta+\tilde{m}_2}{g_{1d}}} + \cdots =$$

$$\frac{\tilde{m}_0 \left(e^{t\psi_6} e^{-t_1\psi_6} - 1\right)}{2 - \frac{\delta}{g_{1d}}} + \frac{\tilde{m}_1 \left(e^{t\psi_6} e^{-t_1(\psi_6 - \tilde{m}_2)} - e^{\tilde{m}_2 t}\right)}{2 - \frac{\delta+\tilde{m}_2}{g_{1d}}} + \cdots =$$

$$-\frac{\tilde{m}_0}{2 - \frac{\delta}{g_{1d}}} + e^{t\psi_6}\left(\frac{\tilde{m}_0 e^{-t_1\psi_6}}{2 - \frac{\delta}{g_{1d}}} + \frac{\tilde{m}_1 \left(e^{-t_1(\psi_6 - \tilde{m}_2)}\right)}{2 - \frac{\delta+\tilde{m}_2}{g_{1d}}} + \frac{\tilde{m}_3 \left(e^{-t_1(\psi_6 - \tilde{m}_4)}\right)}{2 - \frac{\delta+\tilde{m}_4}{g_{1d}}}\right) -$$

$$\left(\frac{\tilde{m}_1 e^{t\tilde{m}_2}}{2 - \frac{\delta+\tilde{m}_2}{g_{1d}}} + \frac{\tilde{m}_3 e^{t\tilde{m}_4}}{2 - \frac{\delta+\tilde{m}_4}{g_{1d}}}\right)$$

As before, set $\tilde{m}_2 = \psi_1, \tilde{m}_4 = \psi_3$, and this will take care of the exponents where t shows up.

$$E^{n0}(t,z) = -\frac{\tilde{m}_0}{2 - \frac{\delta}{g_{1d}}} + e^{t\psi_6}\left(\frac{\tilde{m}_0 e^{-t_1\psi_6}}{2 - \frac{\delta}{g_{1d}}} + \frac{\tilde{m}_1 \left(e^{-t_1(\psi_6 - \psi_1)}\right)}{2 - \frac{\delta+\psi_1}{g_{1d}}} + \frac{\tilde{m}_3 \left(e^{-t_1(\psi_6 - \psi_3)}\right)}{2 - \frac{\delta+\psi_3}{g_{1d}}}\right) - \left(\frac{\tilde{m}_1 e^{t\psi_1}}{2 - \frac{\delta+\psi_1}{g_{1d}}} + \frac{\tilde{m}_3 e^{t\psi_3}}{2 - \frac{\delta+\psi_3}{g_{1d}}}\right)$$

The solution to the unknowns is therefore given by

$$\frac{1}{a_0} - \psi_{-1n} - \frac{a_1}{a_0}\psi_{-1} - \frac{\kappa_e}{a_0}\tilde{m}_0 = -\frac{\tilde{m}_0}{2 - \frac{\delta}{g_{1d}}}$$

$$\frac{a_1}{a_0}\tilde{\psi}_0 + \tilde{\psi}_8 + \frac{\kappa_e}{a_0}\tilde{m}_1 = \frac{\tilde{m}_1}{2 - \frac{\delta+\psi_1}{g_{1d}}}$$

$$\frac{a_1}{a_0}\tilde{\psi}_2 + \tilde{\psi}_4 + \frac{\kappa_e}{a_0}\tilde{m}_3 = \frac{\tilde{m}_3}{2 - \frac{\delta+\psi_3}{g_{1d}}}$$

For this to be an equilibrium, we need the following to hold.

$$-\tilde{\psi}_5 = \frac{\tilde{m}_0 e^{-t_1\psi_6}}{2 - \frac{\delta}{g_{1d}}} + \frac{\tilde{m}_1 \left(e^{-t_1(\psi_6 - \psi_1)}\right)}{2 - \frac{\delta+\psi_1}{g_{1d}}} + \frac{\tilde{m}_3 \left(e^{-t_1(\psi_6 - \psi_3)}\right)}{2 - \frac{\delta+\psi_3}{g_{1d}}}$$

or

$$-\tilde{\psi}_5 = -\left[\frac{1}{a_0} - \psi_{-1n} - \frac{a_1}{a_0}\psi_{-1} - \frac{\kappa_e}{a_0}\tilde{m}_0\right] e^{-t_1\psi_6} +$$
$$\left[\frac{a_1}{a_0}\tilde{\psi}_0 + \tilde{\psi}_8 + \frac{\kappa_e}{a_0}\tilde{m}_1\right] e^{-t_1(\psi_6-\psi_1)} +$$
$$\left[\frac{a_1}{a_0}\tilde{\psi}_2 + \tilde{\psi}_4 + \frac{\kappa_e}{a_0}\tilde{m}_3\right] e^{-t_1(\psi_6-\psi_3)}$$

Clearly this condition is implied by the market clearing condition when $t = t_1$.(just multiply both sides by $e^{t_1\psi_6}$)

Redefining the $\tilde{m}'s$,

$$\tilde{m}_0 = \frac{\frac{a_1}{a_0}\psi_{-1} + \psi_{-1n} - \frac{1}{a_0}}{-\frac{\kappa_e}{a_0} + \frac{1}{2-\frac{\delta}{g_{1d}}}}$$
$$\tilde{m}_1 = \frac{\frac{a_1}{a_0}\tilde{\psi}_0 + \tilde{\psi}_8}{-\frac{\kappa_e}{a_0} + \frac{1}{2-\frac{\delta+\psi_1}{g_{1d}}}}$$
$$\tilde{m}_3 = \frac{\frac{a_1}{a_0}\tilde{\psi}_2 + \tilde{\psi}_4}{-\frac{\kappa_e}{a_0} + \frac{1}{2-\frac{\delta+\psi_3}{g_{1d}}}}$$