Dynamic Multilateral Markets

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Abstract

We study dynamic multilateral markets, in which players' payoffs result from coalitional bargaining. The equilibrium payoffs are computed in stationary market equilibria that we show to exist for any market game. We focus, in particular, on market games with different player types and derive an explicit formula for each type's limit payoff as the bargaining frictions vanish. The limit payoff of a type depends in an intuitive way on the supply and the demand for this type in the market, adjusted by the type-specific bargaining power. We consider our framework as an alternative to the Walrasian price-setting mechanism. When we apply it to the theory of labor markets, we find that the firm size and equilibrium payoffs are determined endogenously, with each worker type being rewarded its marginal product.

Keywords: multilateral bargaining, dynamic markets, labor markets

1. Introduction

Market interactions often involve more than two parties. In some situations these are buyers and sellers who generate different trade surplus depending on the relative size of the demand and supply. In other situations each party's participation is necessary for the completion of the transaction. This is a prevailing feature of markets in the presence of intermediaries such as financial and legal institutions and labor markets. This paper aims to contribute to the growing literature on dynamic markets (cf. [15],[2], [16], and, more recently [5]) by focusing on multilateral bargaining. A price formation mechanism based on strategic interactions is studied as an alternative to the Walrasion auctioneer's search for equilibrium prices. In the context of wage-setting, we also study the implication of multilateral bargaining for the size and organizational design of the firm (Examples 2-5). We further employ multilateral bargaining in the analysis of two-sided markets with more than one market participant on each side (Example 7) and endogenize the size of the bargaining coalition.

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We take [15] as a stepping stone and study a multi-sided market, in which subsets agents meet every period and bargain over the allocation of a surplus that they can produce jointly. The surplus is shared either according to the Nash Bargaining Solution with endogenous threat points or a random player becomes a proposer in an ultimatum game (cf. [14]).

In the case of an unanimous agreement, trade is realized and all agreeing agents trade and leave the market. These agents are then replaced by replica agents with the same endowments, thus, the stock of market participants and the distribution of endowments remain constant throughout the entire game. In case that at least one of the matched agents disagrees, the players from the disagreeing coalition remain in the market and new sets of agents are matched in the next period. When we extend the notion of market equilibrium based on stationary strategies to our dynamic multilateral markets, we show that a market equilibrium exists in all games. Unlike other models of the firm based on multilateral bargaining, e.g., [10] [18], [19], we find that, in equilibrium, each factor of production is rewarded by its marginal product. Thus, the bargaining procedure discussed here may be seen as an alternative to the Walrasian price-setting mechanism, which makes explicit the role of strategic behavior.

In particular, when we study markets with homogenous labor, we find that in equilibrium, worker's relative wage does not depend on bargaining frictions, as embodied in the discount factor and the matching probabilities. Instead, it depends in an intuitive way on the relative bargaining power between the entrepreneur and the worker and it is an increasing function of the relative labor market tightness given by the ratio of total labor supply and the number of vacancies. We further endogenize firm size (the number of vacancies in each period) in the case when the entrepreneur's bargaining power equals the collective bargaining power of the workers. We show that firm size is positively related to the total labor supply and it is negatively related to the concavity of the production function. For the same case when the entrepreneur's bargaining power equals the collective bargaining power of the workers, we also study the effect of (ex-post) unionization. Surprisingly, we find that workers' payoffs are higher when they bargain for their wage individually than when they do so collectively as a trade union.

We focus, then, on market games with heterogeneous players. In particular, we analyze the situation, in which each productive coalition (e.g., a firm) employs a fixed combination of player types. In the general setting, we show that the aggregate payoff of a type depends on its relative bargaining power, relative demand and supply for this type and does not depend on the discount factor and the matching probability. In the case of vanishing bargaining frictions, we derive the unique market equilibrium payoffs that specify this dependence on the individual level of skill supply for each player type.

We use these general results to study a dynamic multilateral market with heterogeneous factors of production that may have asymmetric productivity. Such market features are present in hierarchical firms where a worker's type corresponds to a level in the firm's hierarchy. The results from homogeneous-type-firm generalize nicely to the heterogeneous-type framework. In particular, we generalize the equilibrium wage equation and give it a labor demand-supply interpretation.

Our work differs from other studies on multilateral markets in the bargaining procedure and matching mechanism. Several other works study multilateral bargaining, however, they do not consider a dynamic market. One strand of this literature focuses on games with a characteristic function form. These authors either aim at supporting all core allocations as equilibrium outcomes (cf [22]) or focus on the efficiency of the bargaining outcomes (cf. [3] and [12]). In a related work, [10] allow for "partial agreement" where the agreeing agents leave the bargaining procedure with their agreed shares, while the proposer and the disagreeing agents proceed to the next stage of bargaining. While the assumption of "partial agreement" may be valid in certain contexts, e.g., division of an estate, it is less applicable to others such as production with complementary inputs. [4] instead study unanimous agreement of all parties in the multi-person ultimatum game. They, however, assume that bargaining continues until agreement is reached. As our focus is on anonymous dynamic markets, the assumption that the bargaining coalition dissolves in case of disagreement seems more plausible.

The main application of our general theory of multilateral dynamic market is built on the labor market theory. In this respect, our work is related to [18] and [19]. Our bargaining procedure differs from the one studied by these two author in that it treats all factors of production symmetrically. In particular, whereas the threat points of the workers in [18] are exogenously given, they are determined endogenously in our model. The implications that we derive with regards to wages and firm size differ therefore, not surprisingly, from those of [19]. In this respect, our model can be viewed as providing an alternative wage-setting mechanism to the tâtonnement process of competitive equilibrium.

The remainder of the paper is organized as follows: In Section 2 we describe our theoretical framework and in Section 3 we state our main existence result and apply our general theory to the theory of the firm. In Section 4, we specialize our discussion of market equilibria by introducing heterogeneous player types. In this context, we develop the notion of type-separating market equilibria and compute the equilibrium payoffs explicitly, when the bargaining frictions vanish. Section 6 provides some concluding remarks.

2. Definitions and Market Interaction

We consider a market with the set $\mathcal{N} = \{1, \ldots, N\}$ of replica agents that operates at discrete dates. Each date starts with a matching stage, in which coalitions of players (subsets of $\mathcal{N}$) are randomly matched. We allow for many coalitions (possibly of different sizes) to be matched at the same date as long as all intersections of the matched coalitions are empty set. The probability π_S of selecting the coalition $S \subseteq \mathcal{N}$ is implied by a stationary matching procedure and, hence, is constant throughout the game. When matched, the members of S can produce a surplus $v(S) \geq 0$ by employing their player-specific inputs. We assume that all productive coalitions have a positive probability of meeting, $\pi_S > 0$ if $v(S) > 0$. We assume, further, that either they share this surplus according to the Nash Bargaining Solution (NBS) or that one of the players is chosen as proposer in an ultimatum bargaining game. The (absolute) bargaining power of each player $i \in \mathcal{N}$ will be parameterized by α_i that we assume to be strictly positive. Let the set of all coalitions $S \subseteq \mathcal{N}$ containing player $i \in \mathcal{N}$ be denoted by S_i. Then, the probability that player i is proposer in $S \in S_i$ is given by $\alpha_i/\alpha(S)$, where $\alpha(S) := \sum_{k \in S} \alpha_k$, for all $i \in S$. In the context of Nash

Bargaining, $\alpha_i/\alpha(S)$ is the (relative) bargaining power of i in coalition S. The threat points (or minimum accepted offers) will be determined endogenously in market equilibrium. We note that the assumption of a simultaneous multilateral bargaining in a matched coalition S is a convenient simplification. Our results obtain also when the players in S bargain sequentially, as long as the same player proposes in each bargaining round and we treat all bargaining rounds as occurring in the same period.

As we will show in the next section, a equilibrium disagreement in S will occur whenever the sum of threat points (the sum of minimum acceptance levels) exceeds the trade surplus $v(S)$. In this case, the same set of players proceeds to the next date, where coalitions are selected randomly and the bargaining process starts again. Otherwise, there is an agreement, in which case all agents in S receive their agreed shares of $v(S)$ and leave the market. All players that have left the market are instantly replaced by replica agents with the same endowments. Importantly, all new agents are treated by the matching procedure in the same way as the ones who left. In particular, the set of newcomers that have replaced the members of an agreeing coalition S, will be selected with probability π_S in all ensuing periods, in which it stays in the market. We do not rule out the possibility that players may return to the market some periods after their departure. For our results to hold, it is only important that players do not anticipate strategically this possibility when bargaining over their shares. We further assume that all players apply a common discount factor δ at each stage of the game.

3. Stationary Equilibria in the Market Game

One way to define market equilibria in our game is to follow Rubinstein and Wolinsky (1985) - RW 1985 thereafter - and set up an extensive form game. This definition specifies histories and strategies for all players. In particular, a history of an agent at a certain stage of the game is a sequence of observations made by her up to that stage and. A strategy of an agent is a sequence of decision rules after all histories such that this agent moves, i.e., makes an offer or responds to an offer. RW 1985 focus on stationary strategies, i.e., strategies that prescribe a history-independent bargaining behavior towards the partners with whom an agent is matched. Thus market equilibrium (ME thereafter) is defined as a stationary strategy profile such that no agent can improve by changing her action after any possible history. The formalization of the RW 1985 framework to the market game with multilateral coalitions is straightforward and will be omitted here. Instead, we construct MEa by observing that in such an equilibrium all coalitional partners of player i offer her the discounted continuation payoff δx_i and player i is indifferent between accepting and rejecting this offer. Therefore, the equilibrium payoffs that players expect at the start of each period solve the following system,

$$\begin{aligned} x_i &= \sum_{S\in\mathcal{S}_i} \widetilde{\pi}_S \left(\frac{\alpha_i}{\alpha(S)} \left(v(S) - \delta x(S\backslash\{i\})\right) + \frac{\alpha(S\backslash\{i\})}{\alpha(S)} \delta x_i \right) + (1 - \sum_{S\in\mathcal{S}_i} \widetilde{\pi}_S) \delta x_i \\ &= \delta x_i + \sum_{S\in\mathcal{S}_i} \widetilde{\pi}_S \frac{\alpha_i}{\alpha(S)} (v(S) - \delta x(S)), \quad \forall i \in \mathcal{N}, \end{aligned} \tag{1}$$

where $\widetilde{\pi}_S/\pi_S$ is the agreement probability in the matched coalition S and $x(T) := \sum_{j\in T} x_j$ for all $T \subseteq \mathcal{N}$. A matched coalition S will agree in ME with positive probability only if the agreement is profitable for all members of S, i.e., whenever the sum of discounted continuation payoffs, $\delta x(S)$, does not exceed the productivity $v(S)$. Otherwise, either at least one responder $j \in S$ faces an offer that is below δx_j or the proposer i ends up with the residual surplus $v(S) - \delta x(S\backslash\{i\})$ below her discounted continuation payoff δx_i. It follows that $\widetilde{\pi}_S$ must satisfy,

$$\begin{aligned} \delta x(S) < v(S) &\Rightarrow \widetilde{\pi}_S = \pi_S, \\ \delta x(S) > v(S) &\Rightarrow \widetilde{\pi}_S = 0, \\ \delta x(S) = v(S) &\Rightarrow \widetilde{\pi}_S \in [0, \pi_S], \end{aligned} \tag{2}$$

Formally, a ME for the discount factor $\delta \in [0,1]$ is a solution $(x^\delta, \widetilde{\pi}^\delta)$ to the system (1)-(2), where $x^\delta = (x_i^\delta)_{i\in\mathcal{N}}$ is a feasible allocation and $\widetilde{\pi}^\delta = (\widetilde{\pi}_S^\delta)_{S\subseteq\mathcal{N}}$ contains the probabilities of cooperation. We will say that a coalition S is active in the ME $(x^\delta, \widetilde{\pi}^\delta)$ if $\widetilde{\pi}_S^\delta > 0$, i.e., if the coalition S cooperates with positive probability in this ME.

Note that re-writing (1) as,

$$x_i = (1 - \sum\nolimits_{S\in S_i} \widetilde{\pi}_S)\delta x_i + \sum\nolimits_{S\in S_i} \widetilde{\pi}_S \{\delta x_i + \frac{\alpha_i}{\alpha(S)}(v(S) - \delta x(S))\} \tag{3}$$

makes clear that the expected payoff x_i results also when the outcome of each coalitional meeting is prescribed by the NBS where player's bargaining power is given by $(\alpha_i/\alpha(S))_{i\in S}$ and the (endogenous) threat points are $(\delta x_i)_{i\in S}$.

We can also interpret the payoff x_i as the price that player i expects for her input in a ME. In any case, x_i is different from the expected $i's$ per-period payoff, which is equal to the second term in (3).[1] Note that in an anonymous market games, i.e., one in which all players have the same bargaining power, the value of a coalition depends only on the number of players in that coalition and the matching probability of all coalitions of equal size is the same, the expected per-period equilibrium payoff of each player does not depend on δ as stated in Proposition 1. Formally anonymous markets are characterized by $\alpha_i = \alpha_j$ for all $i, j \in \mathcal{N}$, $v(S) = v(T)$ and $\pi_S = \pi_T$ for all $S, T \subseteq \mathcal{N}$ with $|S| = |T|$.

Proposition 1. *Consider an anonymous market game and a symmetric market equilibrium of that game $(x^\delta, \widetilde{\pi}^\delta)$. The expected per-period payoff is given by $\sum_{S\in S_i} \widetilde{\pi}_S \frac{v(S)}{|S|}$ for all players $i \in \mathcal{N}$. The individual payoff, x_i^δ for all $i \in \mathcal{N}$ in this symmetric equilibrium is given by $\frac{\sum_{S\in S_i} \widetilde{\pi}_S \frac{v(S)}{|S|}}{1-\delta(1-\sum_{S\in S_i} \widetilde{\pi}_S)}$.*

[1] This distinction has important implications, when agents enter the market with a stock of inputs. If agents could trade repeatedly and stay in the market until the stock is exhausted, the disagreement payoffs (1) would not internalize the utility loss due to delayed cooperations. Although we allow each replica agent to trade only once, our model can be easily extended in this direction.

The first part of the statement is straightforward to derive from (3) using that $\frac{\alpha_i}{\alpha(S)} = \frac{1}{|S|}$ and $x(S) = |S|x_i$ for all $S \subseteq \mathcal{N}$ and any $i \in \mathcal{N}$ in a symmetric equilibrium of an anonymous market game. The second part is easily derived by replacing the second term of (3) with $\sum_{S \in S_i} \widetilde{\pi}_S \frac{v(S)}{|S|}$.

The following example is built on an anonymous market game. It is used to illustrate the further point, that unlike in the bargaining models of [8] and [21](the expected) ME payoffs (1) may not be equal to the Shapley value[2] and may not be in the core of the corresponding static game.

Example 1 (Pairwise trade). Let $\mathcal{N} = \{1,2,3\}$ with $\alpha_i = 1/3$ for all $i \in \mathcal{N}$. Let $v(\{i,j\}) = 1$ for all $i,j \in \mathcal{N}$ with $i \neq j$ and $v(S) = 0$, otherwise. Consider a matching procedure $\pi^{\{i,j\}} = 1/3$ for all $i,j \in \mathcal{N}$ with $i \neq j$. Since all players are symmetric, the system (1) simplifies to one equation when all pairs agree,

$$x = \delta x + 2\frac{1}{3}\frac{1}{2}(1 - \delta 2x).$$

The solution $x = \frac{1}{3-\delta}$ entails that $2\delta x \leq 1$ for all $\delta \in [0,1]$ and, hence, the optimal agreement condition (2) holds. The expected per-period equilibrium payoff of each player, $x - \delta x/3 = 1/3$, does not depend on δ as stated in Proposition 1. Moreover, the latter value and x differ from the Shapley value for the productive coalitions, i.e., the coalitions of size two when $\delta \neq 1$[3], and are not in the core of the corresponding static game, which is empty.

Moreover, it turns out that a ME, i.e., a solution to the system (1)-(2), exists in any game.

Proposition 2. *There exists a ME in any game.*

Proof: The proof is relegated to the Appendix.

In the following series of examples, we apply our theoretical framework to the study of the labor market. We derive the equilibrium wage vector and firm size. In addition, we discuss the implications of (ex-post) unionization, i.e., when at the bargaining stage workers act collectively.

Example 2 (Labor Market with Homogeneous Workers). At each date, the labor market consists of one entrepreneur, i, and N homogenous workers. We will denote the set of workers by $\mathcal{N}_w$ and the common worker's bargaining power by α_w. The productivity of a coalition with $n \leq N$ workers and the entrepreneur (a productive coalition) is given by the increasing and concave production function $F(n) : \mathbb{N} \mapsto \mathbb{R}_+$, i.e., $v(S) = F(n)$ for all $S \subseteq \mathcal{N}$ such that $i \in S$ and $|S \cap \mathcal{N}_w| = n$. In particular, production is impossible without workers, $F(0) = 0$.

[2]As Example 7 illustrates there are some market games in which the ME payoffs and the Shapley value do coincide.

[3]The Shapley value for a productive coalition of size two is given by (1/2,1/2).

We can compute the ME payoffs for the entrepreneur and the representative worker, x_i^δ and x_w^δ, respectively, from (1) when only coalitions with n workers are matched. For symmetric matching probabilities, the system (1) specializes, then, to two equations,

$$x_i = \delta x_i + \frac{\alpha_i}{\alpha(n)}(F(n) - \delta x_i - n\delta x_w), \tag{4}$$
$$x_w = \delta x_w + \frac{\binom{N-1}{n-1}}{\binom{N}{n}}\frac{\alpha_w}{\alpha(n)}(F(n) - \delta x_i - (n)\delta x_w)$$

where $\alpha(n) := \alpha_i + n\alpha_w$ and we assume that agreement is optimal in every productive coalition. It can be shown that the solution to the latter system confirms the optimality of agreements and that the ratio of equilibrium payoffs between the entrepreneur and a representative worker satisfies,

$$\frac{x_i^\delta}{x_w^\delta} = \frac{\alpha_i N}{\alpha_w n}, \tag{5}$$

Notably, expression (5) does not depend on the discount factor δ. Instead, it depends in an intuitive way on the relative bargaining power between the entrepreneur and the worker and it is an increasing function of the relative labor market tightness, N/n where N measures total labor supply and n the number vacancies in the firm.

From the solution (x_i^δ, x_w^δ) to 4, we can calculate the limit payoffs,

$$\lim_{\delta\to 1} x_i^\delta = x_i^1 = \frac{\alpha_i F(n) N}{\alpha_i N + \alpha_w n^2}, \tag{6}$$
$$\lim_{\delta\to 1} x_w^\delta = x_w^1 = \frac{\alpha_w F(n) n}{\alpha_i N + \alpha_w n^2},$$

which do not depend on the matching probabilities and are a special case of the limit payoffs in Proposition 3 (Section 4). Note that

$$x_i^1 + n x_w^1 = F(n),$$
$$\partial x_w^1/\partial N < 0, \quad \partial x_i^1/\partial N > 0\ (\leq 0) \quad if \quad F(1n) > 1\ (\ \leq 1).$$

The first equation states that workers and the entrepreneur exhaust the product of their cooperation, while the derivatives in the second line are similar to the findings in search models, e.g. [17]: the tighter the labor market (the lower the N) the higher the worker's wage (x_w^1) is. The reverse is expected to hold with respect to the entrepreneur's payoff.

So far, our assumption that only coalitions with n workers are matched led to the equilibrium payoffs (6). In general, however, all productive coalitions should have a (random) opportunity to cooperate. Interestingly, if we focus on pure strategy, generic limit MEa (GLME)[4], a unique firm

[4] The generic limit ME is formally defined in Definition 12.

size and the unique limit payoffs (6) emerge endogenously. Pure strategy refers here to the absence of randomized coalitional agreements, while genericity rules out a specific relationship between equilibrium payoffs and coalitional productiveness (see definition (12)).

Our Propositions 3 and 4 imply, then, that only coalitions with a particular number, say n, of workers will agree in a GLME $(x^1, \widetilde{\pi}^1)$ and each player type will earn the limit payoff, given by (6). Furthermore, Lemmata 1, 6 imply that cooperation for any other firm size, say with $k \neq n$ workers, is not supported in equilibrium as the coalition's value falls short of the members' total limit payoffs,

$$x_i^1 + kx_w^1 \geq F(k), \quad k = 1, ..., N, \tag{7}$$

with equality for $k = n$ only. In particular, for $k = n - 1$ and $k = n + 1$, the conditions (7) take the form,

$$x_i^i + (n-1)x_w^i > F(n-1) \quad \& \quad x_i^i + (n+1)x_w^i > F(n+1),$$

which, after substituting $x_i^i + nx_w^i = F(n)$ becomes,

$$F(n) - F(n-1) > x_w^i > F(n+1) - F(n). \tag{8}$$

These inequalities define, essentially, the neoclassical wage for the workers.

Example 3 (Equilibrium firm size). In the homogeneous labor market, discussed in Example 2, we derive the limit equilibrium payoffs (6). If the bargaining power of the entrepreneur is the same as the collective bargaining power of the workers, $\alpha_i = \alpha_w n$, the payoffs (6) simplify to,

$$x_i^1 = \frac{F(n)N}{N+n}, \quad x_w^1 = \frac{F(n)}{N+n}. \tag{9}$$

On the other hand, the neoclassical wage condition (8) indicates that the wage should approximate the product of the marginal worker $F'(n)$. In particular, for the concave Cobb-Douglas production function $F(n) = An^\gamma$, $\gamma \in (0, 1)$, we obtain the limit equilibrium firm size as,

$$x_w^1 = \frac{F(n)}{N+n} = F'(n) \Rightarrow n = \frac{N\gamma}{1-\gamma}, \tag{10}$$

where we assume that this value is an integer. We validate this claim by substituting (9) and (10) into (7), which leads to the condition,

$$\frac{p+1}{p}(\frac{p\gamma}{1-\gamma})^\gamma(1-\gamma) \geq 1, \quad where \quad p = N/k. \tag{11}$$

The derivative w.r.t. p of the l.h.s. of this expression is zero for $p^* = (1-\gamma)/\gamma$ and negative (positive) below (above) this value. Hence, the l.h.s. of (11) attains its minimum at p^*, or for $k = Np^* = N\gamma/(1-\gamma)$. For this value, the condition is satisfied with equality, while for all other values of $p = N/k$, it is satisfied with unequality.

Interestingly, we obtained a full characterization of the limit ME[5] - the unique firm size, wage and profit - without reference to an exogenous wage (the outside option of the worker). Notice that equilibrium firm size is larger the greater the total labor supply, N, is. Furthermore the firm size increases in γ, i.e., more concave production functions imply smaller equilibrium firms.

The last example in this section considers the implications of unionization.

Example 4 (Trade unions). We consider the homogeneous labor market from Example 2 with the GLME firm size n. We suppose that after the matching stage, the selected n workers form a trade union. We let the bargaining power of the trade union, $\alpha_u = n\alpha_w$, be equal to the sum of the bargaining powers of its n members. Under these assumptions, the original game, in which the entrepreneur is randomly matched with n workers, is transformed into a bilateral bargaining game, in which the entrepreneur is randomly matched with one of the $\widetilde{N} = \binom{N}{n}$ potential trade unions. The limit payoffs in the transformed game are given in Propositions 3,

$$\widehat{x}_i^1 = \frac{F(n)\alpha_i\widetilde{N}}{\alpha_i\widetilde{N} + \alpha_w n}, \quad \widehat{x}_u^1 = \frac{F(n)n\alpha_w}{\alpha_i\widetilde{N} + \alpha_w n}.$$

for the entrepreneur and the trade union, respectively. When comparing worker's payoffs x_w^1 and $\widehat{x}_u^1/n$ under equal bargaining powers of the entrepreneur and the unionized workers, i.e., $\alpha_i = n\alpha_w$, we obtain - somehow surprisingly - that the ex-post unionization harms the workers (except in the extreme cases of $n = 1$ and $n = N$ when it is inconsequential),

$$\frac{\widehat{x}_u^1/n}{x_w^1} = \frac{\alpha_w F(n)}{\alpha_i\widetilde{N} + \alpha_w n} \Big/ \frac{n\alpha_w F(n)}{\alpha_i N + \alpha_w n^2} = \frac{N+n}{n(\widetilde{N}+1)} \leq 1.$$

4. Limit Market Equilibria with Many Types

In this section, we specialize our concept of market equilibrium to markets with many types of players. We assume that the set of players $\mathcal{N}$ is partitioned into T types, $\mathcal{N} = \cup_{t=1,\ldots,T}\mathcal{N}_t$, $\cap_{t=1,\ldots,T}\mathcal{N}_t = \varnothing$ and $N_t := |\mathcal{N}_t|$. The set of all possible player types is denoted by $\mathcal{T}$. A multilateral coalition (MC) $S \subseteq \mathcal{N}$ consists of $\sum_{t\in S} n_t$ players, where $n_t = |S \cap \mathcal{N}_t|$ denotes the number of t-type players in this coalition. We will often use the operator $T(S)$ to obtain the type of the coalition S (players' type profile). This type is defined as an ordered vector of type multiplicities, $T(S) = (n_1, ..., n_T)$, where $n_t = |S \cap \mathcal{N}_t|$. For example, the type of a one-player coalition is a unit vector, while the type of the grand coalition is $(N_1, ..., N_T)$. Crucially for the following results, we assume that $v(S) = v(S')$ when $T(S) = T(S')$. Therefore, we can use the shorthand $v(T(S))$ for the productivity of a coalition of type $T(S)$. We assume also that players of the same type have equal bargaining powers, $\alpha_i = \alpha_j$ if $T(i) = T(j)$. In order to rule out trivial equilibria, we assume further that $v(S) > 0$ for at least one coalition $S \subseteq \mathcal{N}$.

[5] The limit ME is formally defined in Definition 1.

Definition 1. A ME $(x^1, \widetilde{\pi}^1)$, where $x^1 = \lim_{\delta \to 1} x^\delta$ and x^δ is the solution to (1) for given probabilities $\widetilde{\pi}^1$, will be called a limit ME (LME).

Note that $(x^\delta, \widetilde{\pi}^1)$ is not necessarily a ME in the vicinity of $\delta = 1$ because by keeping the agreement probabilities $\widetilde{\pi}^1$ fixed, some coalitions may be forced to violate the rational agreement condition (2).

Definition 2. A ME $(x^\delta, \widetilde{\pi}^\delta)$ is generic if,

$$\forall S, S' : S \cap S' \neq \varnothing, T(S) \neq T(S'), \quad v(S)x^\delta(S') \neq v(S')x^\delta(S). \tag{12}$$

We will abbreviate generic LME to GLME. In Lemma 6, we prove that a GLME implies that there are no intersecting coalitions of different types, in which agreement is rational simultaneously.

In order to circumvent technicalities, we will focus on pure strategy GLME, i.e., equilibria in which all agreements are non-random.[6]

Proposition 3. *For a pure strategy GLME* (x^1, π^1)*,*

(i) There exists $\delta^* < 1$ *such that* (x^δ, π^1) *is a ME for* $\delta \in (\delta^*, 1]$ *and* $x^1 = \lim_{\delta \to 1} x^\delta$.

(ii) Each player type cooperates in coalitions of homogeneous types.

(iii) All t*-type players, that cooperate in coalitions of type* $\mathbf{n} = (n_1, ..., n_T)$*, receive the same payoff,*

$$x_t^1 = v(\mathbf{n}) \frac{n_t \alpha_t \prod_{s \neq t:} N_s}{\sum_s (n_s^2 \alpha_s \prod_{k \neq s} N_k)}, \tag{13}$$
$$\textstyle\sum_t n_t x_t^1 = v(\mathbf{n}).$$

(iv) There is no other pure strategy GLME with the same payoff vector x^1.

PROOF. The proof follows from the following lemmata.

(i) Lemma 5, (ii) Lemma 3, (iii) Lemma 7, (iv) Lemma 8.

In many games, a productive cooperation requires participation of all player types, i.e., $v(n_1, .., n_T) > 0$ only if $n_t \geq 1$ for all types t. For example, consider an environment in which all factors of productions are perfect complements as it is the case in our application to homogeneous labor markets. For these types of games, we have the following uniqueness result.

Proposition 4. *Consider a market game such that* $v(\mathbf{n}) > 0$ *implies* $\min(\mathbf{n}) \geq 1$*. Then, there is a unique pure strategy GLME.*

Proof. The proof is relegated to the Appendix.

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[6] We can, however, construct a pure strategy GLME that emulates the payoffs in any mixed strategy limit equilibrium.

5. Examples

In our first example we generalize the results derived for homogeneous labor markets.

Example 5 (Labor Market with Heterogeneous Labor). At each date, the labor market consists of one entrepreneur and N_t workers of type $t \in \mathcal{T}$. The production function takes now the form $F(n_1, ..., n_T) : \mathbb{N}^T \mapsto \mathbb{R}_+$, where n_t refers to the number of t-type workers employed by the firm. Alternatively, we can interpret F as the output of a hierarchy with T levels and n_t employees at each level t. We assume further, that productive cooperation is only possible when all types of workers participate, $F(n_1, ..., n_T) > 0$ implies $n_t \geq 1$ for any type t.

From Proposition 4 follows that there is a unique pure strategy GLME $(x^1, \widetilde{\pi}^1)$ in this game. In this equilibrium, only coalitions, composed of the entrepreneur and a particular profile $\mathbf{n} = (n_1, ..., n_T)$ of workers, agree and player types earn the limit payoffs (13) that exhaust the product,

$$\textstyle\sum_{t=1}^{T} n_t x_t^1 = F(\mathbf{n}).$$

In particular, the relative limit wages of different worker types (different levels in a hierarchy) satisfy,

$$\frac{x_t^1}{x_s^1} = \frac{n_t \alpha_t N_s}{n_s \alpha_s N_t}, \quad s, t \in \mathcal{T}. \tag{14}$$

The latter relationship reveals that the relative scarcity of types in the market influences the relative wages in the expected way: A higher supply N_t of type t workers depresses their relative individual wages ceteris paribus. On the other hand, the total number n_t of workers of type t, employed in an equilibrium firm, has a positive impact on their relative wages. This is not surprising, when we observe that n_t represents the demand for this type by the equilibrium firm. Therefore, (14) combines in a clear-cut manner the supply and demand forces in the market, adjusted by the type-specific bargaining powers).

Furthermore, iterating the argument from Example 2 over types, it is easy to show that the neoclassical wage holds for each type $t \in \mathcal{T}$ in this equilibrium,

$$F(n_t, n_{-t}) - F(n_t - 1, n_{-t}) > x_t^1 > F(n_t + 1, n_{-t}) - F(n_t, n_{-t}), \quad t \in \mathcal{T}. \tag{15}$$

Our next example generalizes two-sided markets.

Example 6 (N-Sided Markets). In an N-sided market, the value of a coalition of type $\mathbf{n} = (n_1, .., n_N)$, where n_i is the number of agents of type i, is given by the Cobb-Douglas function $v(\mathbf{n}) = An_1^{\beta_1} ... n_N^{\beta_N}$, $\beta_i > 0$. Note that the special case $N = 2$ and $n_1, n_2 \leq 1$ corresponds to a homogeneous "buyer-seller" market and the LME payoffs follow, then, from (13),

$$x_1^1 = \frac{\alpha_t N_2}{\alpha_1 N_2 + \alpha_2 N_1}, \quad x_2^1 = \frac{\alpha_2 N_1}{\alpha_1 N_2 + \alpha_2 N_1}.$$

Generally, Proposition 4 implies that there is a unique pure strategy GLME $(x^1, \widetilde{\pi}^1)$ in this game. In this equilibrium, only coalitions of a particular profile $\mathbf{n} = (n_1, ..., n_T)$ cooperate. By a similar argument as used in previous examples, we can approximate the limit GLME payoff x_t^1 by the marginal contribution of this type,

$$\frac{\partial v(\mathbf{n})}{\partial n_t} = \frac{\beta_t}{n_t} v(\mathbf{n}),$$

which, together with (13), leads to,

$$\frac{\beta_t}{n_t} v(\mathbf{n}) = x_t^1 \Rightarrow \frac{n_t^2}{n_s^2} = \frac{\beta_t \alpha_s N_t}{\beta_s \alpha_t N_s}.$$

We assume, without the loss of generality, that in this GLME $n_1 \leq n_s$ for all $s = 1, ..., N$. Then,

$$n_t = (\frac{\beta_t \alpha_s N_t}{\beta_s \alpha_t N_s})^{1/2}, \quad t = 1, ..., N,$$

defines the single coalition type that cooperates in the unique GLME if $\sum_s \beta_s < 1$. After substituting the last expression into (13), we can compute the ratio of total GLME payoffs for the types t and s,

$$\frac{n_t x_t}{n_s x_s} = \frac{\beta_t}{\beta_s}.$$

Finally, we show in the last example that GLME payoffs are sometimes more plausible than, e.g., the Shapley values.

Example 7. Each player $\{1, ..., N\}$ is indexed by her marginal productivity, i.e., player i increases the productivity of any coalition by i. Hence, the value of a coalition is the sum of the marginal productivity of its members. It can be verified that $x^1 = (1, ..., N)$ is a LME payoff vector (this LME is, however, not a GLME. Each player obtains in this LME her marginal productivity. The LME payoffs in this case are different from Shapley values, as can be seen, e.g., for $N = 3$, where $sh = (3/2, 2, 5/2)$.

6. Conclusion

We have developed a price-setting mechanism that makes explicit the role of strategic behavior in the context of dynamic multilateral markets. We have shown the existence of market equilibria in stationary strategies in any multilateral market game. Furthermore, we have studied in more detail the implications of our theoretical framework for the equilibrium price in the labor market. Unlike, other price-setting mechanisms based on multilateral bargaining applied to the labor market, we find that our procedure results in equilibrium prices which equal the respective marginal product of the factors of production. In this respect, we see our model as providing a strategy-based microeconomic alternative to the Walrasian auctioneer's procedure to find the competitive prices.

Further extensions to the current study of the labor market are possible. Similar to [18] one can address the question of technology choice in the context of the type separating market equilibria and investigate how the interplay of factors' productivity and market tightness shape up the equilibrium outcome. In addition, one can explore further the implications of our theoretical framework with respect to organizational design of the firm. The hierarchical depth and width of the firm may be derived endogenously in market equilibrium as a function of the relative bargaining powers and the scarcities of the factors of production.

Last, we want to stress the greater applicability of our general theory to topics beyond the one of the labor market. Many two-sided markets such as the housing market, credit-card market, retailing exhibit similar multilateral structure of market interactions.

7. Appendix

PROOF. *Proposition 2*:

In a stationary ME, all trading partners of i will offer her minimum accepted offer δx_i. Hence, the ME payoffs result from the solution to the system of linear equations for $x = (x_i)_{i \in \mathcal{N}}$,

$$x_i = \delta x_i + \textstyle\sum_{S \in \mathcal{S}_i} \widetilde{\pi}_S \frac{\alpha_i}{\alpha(S)} (v(S) - \delta x(S)), \quad \forall i \in \mathcal{N}. \tag{16}$$

The acceptance probability $\widetilde{\pi}_S / \widetilde{\pi}_S$ will be positive only if a transaction is profitable for all matched players in S, i.e., when $\delta x(S) \leq v(S)$. Otherwise, by offering the minimum acceptable offer δx_i to each player in $S \backslash j$, the proposer j would obtain $v(S) - \delta x(S \backslash j) < \delta x_j$, i.e., less than her discounted expected payoff in the next period. Hence,

$$\begin{aligned} &\delta x(S) < v(S) \Rightarrow \widetilde{\pi}_S = \pi_S, \\ &\delta x(S) > v(S) \Rightarrow \widetilde{\pi}_S = 0, \\ &\delta x(S) = v(S) \Rightarrow \widetilde{\pi}_S \in [0, \pi_S], \end{aligned} \tag{17}$$

A stationary ME is then defined as a solution $(x^\delta, \widetilde{\pi}^\delta) \in [0,1]^N \times [0,1]^{2^N}$ to the system (16)-(17). If we re-write (16) in the matrix from,

$$x = Ax + b, \quad A = (a_{ij}),$$

with the suitable defined matrix A, then it can be readily verified that $||A||_1 = \max_i \sum_j |a_{ij}| = \delta$, which implies that (16) is a contraction with a unique solution for $\delta < 1$. In this case, we define the correspondence $\Lambda(\widetilde{\pi}) : [0,1]^{2^N} \rightrightarrows [0,1]^{2^N}$, which first computes the solution x to (16) for $\widetilde{\pi} = \{\widetilde{\pi}_S\}_{S \subseteq \mathcal{N}}$ and then selects $\widetilde{\pi}'$ from (17) for x. As this correspondence is u.h.c., a fixed point exists.

For $\delta = 1$, we show in Lemma 1 that,

$$v(S) \leq x^1(S), \quad \forall S \subseteq \mathcal{N},$$

with equality for active coalitions. The existence of a solution to the latter system of inequalities follows from the existence of a solution to the constrained linear optimization,

$$\min_{x \in R_+^N} x(\mathcal{N}) \quad s.t. \quad v(S) \leq x(S), \quad \forall S \subseteq \mathcal{N}.$$

Obviously, there is a $x^* \in R_+^N$ that satisfies all inequalities, i.e., $v(S) \leq x^*(S), \forall S \subseteq \mathcal{N}$. Furthermore, for each $i \in \mathcal{N}$, there exists a subset $S^i \subseteq \mathcal{N}$ such that $v(S^i) = x_i^* + x^*(S^i/\{i\})$ if $x_i^* > 0$. Otherwise, x_i^* could be lowered until meeting the equality. Setting $\widetilde{\pi}^1_{S^i} = \pi_{S^i} > 0$ completes the construction of the ME for $\delta = 1$.

Lemma 1. *In a ME $(x^1, \widetilde{\pi}^1)$, it holds for any $S \subseteq \mathcal{N}$ that $x^1(S) \geq v(S)$ (with equality if $\widetilde{\pi}^1_S > 0$).*

PROOF. The system (1) for the ME $(x^1, \widetilde{\pi}^1)$ reads,

$$0 = \textstyle\sum_{S \in S_i} \widetilde{\pi}^1 \dfrac{\alpha_i}{\alpha(S)}(v(S) - x^1(S)), \quad \forall i \in \mathcal{N}. \tag{18}$$

which rules out that $v(S) > x(S)$ for any coalition S. Otherwise, (18) would be a contradiction as $\widetilde{\pi}^1_S = \pi^1_S > 0$ if $v(S) > x^1(S) \geq 0$ and $\widetilde{\pi}^1_S = 0$ if $x^1(S) > v(S)$ by the rational agreement conditions (2). Equation (18) implies, in particular, that $v(S) = x^1(S)$ if $\widetilde{\pi}^1_S > 0$.

Lemma 2. *In a ME $(x^1, \widetilde{\pi}^1)$,*

$$x^1(S) = x^1(S') = \textstyle\sum_{t=1}^T n_t x^1_t, \quad \forall S, S' \subseteq \mathcal{N}, T(S) = T(S') = (n_1, ..., n_T).$$

PROOF. We start with $S = \{i\}$, $S' = \{j\}$, $T(i) = T(j)$, and assume, for the sake of contradiction, that $x^1_i > x^1_j \geq 0$. As $x^1_i > 0$, it follows from the last condition in (2) and from Lemma 1 that there is an active coalition $S^i := \{i\} \cup C$, such that $v(S^i) = x^1(C) + x^1_i > 0$. If we replace agent i by agent j in S^i, we obtain the coalition $S^j := \{j\} \cup C$ of the same type, $T(S^j) = T(S^i)$. But then, equal productiveness of the two coalitions, $v(S^i) = v(S^j)$, and Lemma 1 imply a contradiction,

$$v(S^i) = x^1(C) + x^1_i > x^1(C) + x^1_j \geq v(S^j) = v(S^i).$$

Therefore, $x^1_i = x^1_j = x^1_t$ for $T(i) = T(j) = t$. This result implies that, in general,

$$x^1(S) = \textstyle\sum_{t=1}^T n_t x^1_t = x^1(S'), \quad \forall S, S' : T(S') = T(S) = (n_1, ..., n_T).$$

Lemma 3. *In a GLME, each player type cooperates in coalitions of homogeneous types.*

PROOF. For the sake of contradiction assume that in the GLME $(x^1, \widetilde{\pi}^1)$, $t-$type players i and j cooperate in coalitions S and S', respectively, with $T(S) = (n_1, ..., n_T) \neq (n'_1, ..., n'_T) = T(S')$. Let the coalition S'' be the same as S' except for the player j that we replace with i. Hence, $T(S) \neq T(S') = T(S'')$ and $i \in S \cap S''$. As S and S' are active, $\widetilde{\pi}^1_S > 0$ and $\widetilde{\pi}^1_{S'} > 0$. Then, Lemmata 1, 2 and $T(S\prime) = T(S'')$ imply,

$$v(S) = x^1(S), \quad x^1(S') = v(S') = v(S'') = x^1(S'').$$

By Lemma 6, $v(S) = x^1(S)$ and $v(S'') = x^1(S'')$ contradict the genericity of $(x^1, \widetilde{\pi}^1)$ as $S \cap S'' \neq \varnothing$ and $T(S) \neq T(S'')$.

Lemma 4. *If the probabilities $\widetilde{\pi}^\delta$ in a ME $(x^\delta, \widetilde{\pi}^\delta)$ for $\delta < 1$ imply that all $t-$type and all $s-$type players cooperate only in coalitions of type $\mathbf{n} = (n_1, ..., n_T)$, then,*

$$n_s \alpha_s x^\delta(\mathcal{N}_t) = n_t \alpha_t x^\delta(\mathcal{N}_s).$$

PROOF. By summing up (1) over all t-type players, one obtains the total payoff of this type when $\delta < 1$,

$$\begin{aligned} x^\delta(\mathcal{N}_t) &= \delta x^\delta(\mathcal{N}_t) + n_t\alpha_t \sum\nolimits_{S:T(S)=\mathbf{n}} \widetilde{\pi}_S \frac{v(S) - \delta x^\delta(S)}{\alpha(S)} \\ &= \frac{n_t\alpha_t}{1-\delta} \sum\nolimits_{S:T(S)=\mathbf{n}} \widetilde{\pi}_S \frac{v(S) - \delta x^\delta(S)}{\alpha(S)} =: \frac{n_t\alpha_t}{1-\delta} \Delta^\delta(\mathbf{n}). \end{aligned}$$

Note that we used the fact that the bargaining power α_t is the same across all players of type t. By the same argument, the total payoff to the $s-$type players is $x^\delta(\mathcal{N}_s) = n_s\alpha_s\Delta^\delta(\mathbf{n})/(1-\delta)$ and, hence, the claim follows.

Lemma 5. *For a pure strategy GLME $(x^1, \widetilde{\pi}^1)$, it holds that $(x^\delta, \widetilde{\pi}^1)$ is a ME for δ sufficiently close to one and $x^1 = \lim_{\delta\to 1} x^\delta$.*

PROOF. In a pure strategy GLME $(x^1, \widetilde{\pi}^1)$, $x^1(S) \geq v(S)$ for any coalition S by Lemma 1. For a coalition S such that $x^1(S) = v(S)$, Lemma 6 and the continuity of the solution $(x^\delta, \widetilde{\pi}^1)$ with respect to δ imply that,

$$\forall S' : T(S) \neq T(S'), S \cap S' \neq \varnothing, \quad x^1(S) = v(S) \Rightarrow x^\delta(S') > v(S'),$$

for δ sufficiently close to one. For such δ, members of S will not cooperate in any coalition S' with type $T(S') \neq T(S)$. If $x^\delta(S) > v(S)$, these members would not cooperate in any coalition of type $T(S)$ either, which leads to the contradiction,

$$0 = x^\delta(S) > v(S) \geq 0.$$

As $(x^1, \widetilde{\pi}^1)$ is a pure strategy GLME, we conclude that $x^\delta \leq v(S)$ and $\widetilde{\pi}^1_S = \pi_S > 0$ for each coalition S such that $x^1(S) = v(S)$.

On the other hand, if $x^1(S') > v(S')$ then $x^\delta(S') > v(S')$ for δ sufficiently close to one due to the continuity of the solution $(x^\delta, \widetilde{\pi}^1)$ and $\widetilde{\pi}^1_S = 0$ is the only rational choice for the members of S.

Hence, there exists $\delta^* < 1$ such that $(x^\delta, \widetilde{\pi}^1)$ is a pure strategy ME for $\delta \in (\delta^*, 1]$. The vector x^δ converges to x^1 (for the fixed $\widetilde{\pi}^1$) by the definition of GLME.

Lemma 6. *In a GLME* $(x^1, \widetilde{\pi}^1)$,

$$\forall S, S' : T(S) \neq T(S'), S \cap S' \neq \varnothing, \quad x^1(S) = v(S) \Rightarrow x^1(S') > v(S').$$

PROOF. The case $x^1(S) = v(S) \Rightarrow x^1(S') < v(S')$ is ruled out by Lemma 1. The case $x^1(S) = v(S) \Rightarrow x^1(S') = v(S')$ implies $x^1(S)v(S') = x^1(S')v(S)$. Then, we have to consider three exhaustive alternatives:

$$v(S)v(S') > 0 \Rightarrow \frac{x^1(S)}{v(S)} = \frac{x^1(S')}{v(S')} \Rightarrow x^1(S)v(S') = x^1(S')v(S),$$
$$v(S) \geq v(S') = 0 \Rightarrow x^1(S) \geq x^1(S') = 0 \Rightarrow x^1(S)v(S') = x^1(S')v(S) = 0,$$
$$v(S') \geq v(S) = 0 \Rightarrow x^1(S') \geq x^1(S) = 0 \Rightarrow x^1(S)v(S') = x^1(S')v(S) = 0,$$

all of which contradict the genericity of $(x^1, \widetilde{\pi}^1)$.

Lemma 7. *The GLME* $(x^1, \widetilde{\pi}^1)$ *payoff to a player of type* t *that cooperates in coalitions of type* $\mathbf{n} = (n_1, ..., n_T)$,

$$x_t^1 = v(\mathbf{n}) \frac{n_t \alpha_t \prod_{s \neq t} N_s}{\sum_s (n_s^2 \alpha_s \prod_{k \neq s} N_k)}, \quad \sum\nolimits_{t=1}^{T} n_t x_t^1 = v(\mathbf{n}).$$

PROOF. By Lemma 3, each player type t cooperates in coalitions of homogeneous types. By Lemma 4, the total payoff for all $t-$type and all $s-$type players, that cooperate in coalitions of the same type $\mathbf{n} = (n_1, ..., n_T)$, satisfy for $\delta < 1$,

$$n_s \alpha_s x^\delta(\mathcal{N}_t) = n_t \alpha_t x^\delta(\mathcal{N}_s).$$

This equality must hold also for the GLME $(x^1, \widetilde{\pi}^1)$ due to the continuity of (1). By Lemma 2, the last equality becomes then,

$$n_s \alpha_s N_t x_t^1 = n_t \alpha_t N_s x_s^1.$$

In particular, $x_t^1 = 0$ implies $x_s^1 = 0$ for any two types that cooperate in a coalition $S : T(S) = \mathbf{n}$. This is only possible in a GLME if $v(S) = v(\mathbf{n}) = 0$. We assume below $x_t^1 > 0$ for type t that cooperates in $S : T(S) = \mathbf{n}$. Then, $\widetilde{\pi}_S > 0$ and,

$$v(S) = v(\mathbf{n}) = x^1(S) = \sum\nolimits_s n_s x_s^1$$

by Lemmata 1 and 2. We divide the last equation by x_t^1 and obtain the GLME payoff of type t,

$$v(\mathbf{n})/x_t^1 = \sum\nolimits_s n_s x_s^1 / x_t^1 = \sum\nolimits_s \frac{n_s^2 \alpha_s N_t}{n_t \alpha_t N_s} \Rightarrow$$
$$x_t^1 = v(\mathbf{n}) / \sum\nolimits_s \frac{n_s^2 \alpha_s N_t}{n_t \alpha_t N_s} = v(\mathbf{n}) \frac{n_t \alpha_t \prod_{s \neq t} N_s}{\sum_s (n_s^2 \alpha_s \prod_{k \neq s} N_k)}.$$

Lemma 8. *There is at most one pure strategy GLME with a given payoff vector.*

PROOF. For the sake of contradiction assume that there are two different pure strategy GLMEa with the same payoff vector x^1, $(x^1, \widetilde{\pi}^1)$ and $(x^1, \widehat{\pi}^1)$, $\widetilde{\pi}^1 \neq \widehat{\pi}^1$. Therefore, there must exist coalitions S and S', $T(S) \neq T(S')$, $S \cap S' \neq \varnothing$ such that S cooperates only in the first and S' cooperates only in the second GLME, i.e., $\widetilde{\pi}^1_S = \pi^1_S > 0$, $\widetilde{\pi}^1_{S'} = 0$ and $\widehat{\pi}^1_S = 0$, $\widehat{\pi}^1_{S'} = \pi^1_{S'} > 0$. But this contradicts Lemmata 1 and 6 as,

$$\widetilde{\pi}^1_S = \pi^1_S > 0 \Rightarrow x^1(S) = v(S) \Rightarrow x^1(S') > v(S') \Rightarrow \widehat{\pi}^1_{S'} = 0.$$

PROOF. Proposition 4:

For the sake of contradiction, we assume that there are two different pure strategy GLMEa with payoff vectors $x^1 \neq y^1$ ($x^1 = y^1$ is ruled out by Lemma 8). First, we note that Lemma 6 implies that only one coalition type, $\mathbf{n}^x = (n^x_1, ..., n^x_T)$ and $\mathbf{n}^y = (n^y_1, ..., n^y_T)$, cooperates in the respective GLME.

Then, the following conditions (i) and (ii) ensure that y^1 and x^1 form part of the respective GLME,

$$(i) \quad \textstyle\sum_{t=1}^T n^x_t x^1_t \leq \sum_{t=1}^T n^x_t y^1_t, \quad (ii) \quad \sum_{t=1}^T n^y_t y^1_t \leq \sum_{t=1}^T n^y_t x^1_t.$$

A violation of (i), for example, implies that coalitions of type $\mathbf{n}^x$ would always agree and earn a higher total payoff than coalitions of type $\mathbf{n}^y$, which invalidates the GLME y^1. By a similar argument, (ii) is a necessary condition for the GLME x^1.

Then, by Lemma 7,

$$x^1_t = \frac{n^x_t \alpha_t N_s}{n^x_s \alpha_s N_t} x^1_s > 0, \quad y^1_t = \frac{n^y_t \alpha_t N_s}{n^y_s \alpha_s N_t} y^1_s > 0, \quad s, t = 1, ..., T,$$

and the conditions (i) and (ii) simplify to

$$\begin{aligned} (i') \quad & \frac{x^1_s N_s}{n^x_s \alpha_s} \sum\nolimits_{t=1}^T \frac{(n^x_t)^2 \alpha_t}{N_t} \leq \frac{y^1_s N_s}{n^x_s \alpha_s} \sum\nolimits_{t=1}^T \frac{(n^x_t)^2 \alpha_t}{N_t} \Rightarrow x^1_s \leq y^1_s, \\ (ii') \quad & \frac{y^1_s N_s}{n^y_s \alpha_s} \sum\nolimits_{t=1}^T \frac{(n^y_t)^2 \alpha_t}{N_t} \leq \frac{x^1_s N_s}{n^y_s \alpha_s} \sum\nolimits_{t=1}^T \frac{(n^y_t)^2 \alpha_t}{N_t} \Rightarrow y^1_s \leq x^1_s. \end{aligned}$$

Hence, $x^1_s = y^1_s$ for any $s = 1, ..., T$, which contradicts $x^1 \neq y^1$.

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